UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2017/2018

BASS IV, B.Ed (Sec.) IV, B.Sc IV

Title of Paper : NUMERICAL ANALYSIS II

Course Number : M411

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Write down the normal equations for approximating a function $f(x)$ using continuous least squares approximation derived using a general polynomial of the form $P_n(x) = \sum_{k=0}^{n} a_k x^k$ [3 marks]

(b) i. Define the Lipschitz condition for solving an initial value problem $y' = f(t, y)$. [3 marks]

ii. Does the function $f(t, y) = \sqrt{t^2 + y^2}$ satisfy the Lipschitz condition. [7 marks]

(c) i. Derive the Taylor method of order 3 that can be used to solve the IVP [4 marks]

$$y' = 2ty, \quad y(0) = 2$$

ii. Use the 3rd order Taylor method with step size $h = 0.2$ to obtain the approximate solution of the IVP at $t = 0.4$ [4 marks]

(d) Consider the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + bu_x$$

where b is a constant. Discretize the equation using the backward difference scheme in time and the central difference scheme in space. [4 marks]

(e) i. Use the method of undetermined coefficients to derive the following 2-Step Adams-Bashforth Explicit method [6 marks]

$$y_0 = \alpha, \quad y_1 = \alpha_1$$
$$y_{i+1} = y_i + \frac{h}{2}[3f(t_i, y_i) - f(t_{i-1}, y_{i-1})]$$

ii. Investigate the consistency and stability of the 2-step Adams-Bashforth method [6 marks]

(f) Write down an $0(h^2)$ finite difference scheme for the following boundary value problem:

$$-\frac{d^2u}{dx^2} + c(x)u = f(x), \quad 0 \le x \le 1.$$
$$u(0) = \alpha, \quad u(1) = \beta$$

where $c(x) \ge 0$, $f(x)$ are given continuous functions in the interval $[0, 1]$ and α and β are known boundary values of $u(x)$. [3 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) State the condition(s) that that must be satisfied for a solution of an initial value problem to exist. [3 marks]

(b) State the condition(s) that that must be satisfied for a solution of an initial value problem to be unique. [3 marks]

(c) What type of initial value problems is well-posed? [3 marks]

(d) Suppose that the perturbation $\delta(t)$ is proportional to t, that is $\delta(t) = \delta t$ for some constant δ. Show *directly* using the associated perturbed problem that the following IVP is well-poised. [11 marks]

$$y' = y + 1, \quad 0 \leq t \leq 1, \quad y(0) = 1$$

QUESTION B3 [*20 Marks*]

B3 (a) Derive the normal equations for continuous least square approximation using a linear function $ax + b$ [8 marks]

(b) Use the normal equations derived in (a) above to find the least squares linear function that best fits the curve $y = \sqrt{2x+1}$ over the interval $0 \leq x \leq \frac{3}{2}$. [4 marks]

(c) Use the Gram-Schmidt process to construct the Legendre polynomials

$$\phi_1(x), \phi_2(x), \phi_3(x)$$

Here, $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x)$ is an orthogonal set on $[-1, 1]$ with respect to the weight $w(x) = 1$, given that $\phi_0(x) = 1$. [8 marks]

QUESTION B4 [*20 Marks*]

B4 (a) Use the Von Neumann analysis to obtain a restriction on the time step size that guarantees stability of the finite difference solution, based on forward difference in time and central difference in space, for solving the differential equation $u_t = bu_x$ ($b > 0$ is a constant) [10 marks]

(b) Consider the boundary value problem

$$y'' = xy' - 3y + e^x, \quad 0 \leq x \leq 1$$

$$y(0) = 1, \quad y(1) = 2.$$

By replacing y' and y'' by central difference quotients, show that the general discretization on 5 sub-intervals gives the following matrix equation [10 marks]

$$\begin{bmatrix} -1.88 & 0.98 & 0 & 0 \\ 1.04 & -1.88 & 0.96 & 0 \\ 0 & 1.06 & -1.88 & 0.94 \\ 0 & 0 & 1.08 & -1.88 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -0.9711 \\ 0.0597 \\ 0.0729 \\ -1.7510 \end{bmatrix}$$

QUESTION B5 [*20 Marks*]

B5 Consider the initial value problem

$$y' = y - t + 1, \quad y(0) = 1$$

with exact solution $y(t) = e^t + t$. Use $h = 0.1$ and the exact solution to get the required starting values wherever necessary, to solve the IVP and estimate $y(0.2)$ using

(a) Taylor method of order 2 [6 marks]

(b) Runge-Kutta method of order 4 [6 marks]

(c) Heun's method [6 marks]

(d) 2-Step Adams-Bashforth Explicit method [2 marks]

$$y_0 = \alpha, \quad y_1 = \alpha_1$$
$$y_{i+1} = y_i + \frac{h}{2}[3f(t_i, y_i) - f(t_{i-1}, y_{i-1})]$$

QUESTION B6 [*20 Marks*]

B6 Consider the initial value problem

(a) Derive the local truncation error for the following Adams-Moulton two-step method [8 marks]

$$y_0 = \alpha, \quad y_1 = \alpha_1$$
$$y_{i+1} = y_i + \frac{h}{12}\left[5f(t_{i+1}, y_{i+1}) + 8f(t_i, y_i) - f(t_{i-1}, y_{i-1})\right]$$

(b) Find the local truncation error for solving the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

using the forward difference in time and central difference in space. [12 marks]

END OF EXAMINATION