UNIVERSITY OF SWAZILAND

FINAL EXAMINATION, 2017/2018

B.Sc. IV, BASS IV, BED. IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:

   a. SECTION A (40 MARKS)
      - Answer ALL questions in Section A.

   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer ANY THREE (3) questions in Section B.
      - If you answer more than three (3) questions in Section B, only the first three questions answered in Section B will be marked.

2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [40 MARKS]: ANSWER ALL QUESTIONS

QUESTION A1

(a) Give an example of a ring satisfying the following given conditions. (Do not prove anything)

(i) A ring with zero divisors
(ii) A ring without unity
(iii) A ring that is not a division ring

(6 marks)

(b) A ring R is a boolean ring if $a^2 = a \forall a \in R$. Show that every boolean ring is abelian.

(6 marks)

(c) Prove that if D is an integral domain, then $D[x]$ is also an integral domain.

(8 marks)

QUESTION A2

(a) Let R be a commutative ring. Show that

$$Na = \{x \in R: ax = 0\}$$

is an ideal of R. (Assume the group properties) (4 marks)

(b) Decide the irreducibility or otherwise of

(i) $x^3 - 7x^2 + 3x + 3 \in \mathbb{Q}[x]$

(ii) $8x^3 + 6x^2 - 9x + 24 \in \mathbb{Q}[x]$

(8 marks)

(c) Show that for a field F, the set of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \text{ for } a, b \in F$$

is a right ideal but not a left ideal of $M_2(F)$ (8 marks)

SECTION B: ANSWER ANY THREE QUESTIONS

QUESTION B3

(a) Describe al units in each of the following rings:

(i) $\mathbb{Z}_7$

(ii) $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$. (4 marks)

(b) Write $x^3 + 3x^2 + 3x + 4 \in Z_5[x]$ as a product of irreducible polynomials (6 marks)

(c) State the Eisentein irreducibility criterion and use it to prove that if p is prime, then the cyclotomic polynomial

$$f(x) = \frac{x^p - 1}{x - 1}$$

is irreducible. (10 marks)

QUESTION B4

(a) Use Fermat's theorem to compute the remainder when 8^{103} is divided by 13. (5 marks)

(b) Prove that every finite integral domain is a field. (5 marks)

(c) Show that the mapping $\phi : \mathbb{C} \to M_2(\mathbb{R})$ given by

$$(a+ib)\phi = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad a, b \in \mathbb{R}$$

is a ring homomorphism. Find its kernel. (6 marks)

(d) State Kroneckers theorem. (Do not prove) (4 marks)

QUESTION B5

(a) Show that $1 + x + x^2 \in \mathbb{Z}_2[x]$ is an irreducible polynomial over $\mathbb{Z}_2$. List all the four elements of the field $Z_2[x]/<1 + x + x^2>$ in the form $a + b\alpha$ with $a, b \in \mathbb{Z}_2$, where α is a root of $1 + x + x^2$ in the extension field of $\mathbb{Z}_2[x]$. Construct Caley tables for addition and multiplication, showing all computations.

(10 marks)

(b) For each of the given algebraic numbers $\alpha \in \mathbb{C}$ find $irr(\alpha, \mathbb{Q})$ and $\deg(\mathbb{Q})$.

(i) $\sqrt{3 - \sqrt{6}}$. (3 marks)

(ii) $\sqrt{\frac{1}{3} + \sqrt{7}}$. (4 marks)

(iii) $\sqrt{2} + i$. (3 marks)

QUESTION B6

(a) Given that every element β of $E = F(\alpha)$ can be uniquely expressed in the form

$$\beta = b_0 + b_1\alpha + b_2\alpha^2 + \ldots + b_{n-1}\alpha^{n-1}$$

where each of $b_i \in F$, α algebraic over the field F and $\deg(\alpha, F) = n \geq 1$. Show that if F is finite with q elements, then $E = F(\alpha)$ has q^n elements.

(6 marks)

(b) Prove that every finite integral domain is a field. (6 marks)

(c) Find the greatest common divisor $d(x)$ of the polynomials:

$f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2$ and
$g(x) = x^3 + 4x^2 + 7x + 4$ over $\mathbb{Q}$ and express $d(x)$ as a linear combination of $f(x)$ and $g(x)$

QUESTION B7

(a) Prove that if R is a ring with unity and N is an ideal of R containing a unit, then $N = R$. (6 marks)

(b) Find all ideals and maximal ideals $\mathbb{Z}_{18}$. (4 marks)

(c) In a ring $\mathbb{Z}_n$ show that

(i) divisors of zero are those elements that are **NOT** relatively prime to n. (5 marks)

(ii) elements that are relatively prime to n cannot be zero divisors. (5 marks)