University of Swaziland

Final Examination, 2017/2018

B.Sc. IV/ B.Ed IV/ BASS IV

Title of Paper : Metric Spaces

Course Number : M431

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked**.
2. Show all your working.

Special Requirements: None

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: Answer ALL Questions

A1. (a) Define a Metric space (M.S.)

(b) and thus prove the generalized triangular inequality for M.S. (3,3)

A2. In a M.S (X,d) $X = R^2$, $x = (-1,1), y = (2,2)$. Find a distance $d(x,y)$ in the following M.S.

(a) Euclidean,

(b) Chicago,

(c) New York (3,3,3)

A3. Let $X = R^2$ and $d(x,y) = min(|x_1 - y_1|, |x_2 - y_2|)$, where $x = (x_1, x_2) \subset X, y = (y_1, y_2) \in X$.

Is (X,d) a M.S.? Explain. (4)

A4. Let the sequence x_n in a set X with the discrete metric converges to $a \subset X$. What can you say about the terms of this sequence? (4)

A5. Define a uniform convergence of real-valued functions on $D \subset R$. (3)

A6. Prove that if for the sequence x_n in M.S. $\lim\limits_{n\to\infty} x_n = x$ and $x \in X$, then any subsequence converges to the same limit. (4)

A7. Give a definition and example of

(a) Bounded subset of M.S.

(b) Diameter of a non-empty bounded subset of M.S.

(c) Two Lipschitz equivalent metrics on a set X. (3,3,4)

SECTION B: Answer any *THREE* Questions

QUESTION B1 [*20 Marks*]

B1. (a) Let $X = R^2$

(i) Define an Euclidean distance $d(x,y)$, where $x, y, \in X$.

(ii) Show that (X,d) is M.S. (2,5)

(b) Given a sequence $x_n(t) = t^n, t \in [0,1]$

(i) Sketch the graphs of $x_n(t)$ for some n.

(ii) Find a limit function $x(t) = \lim_{n\to\infty} x_n(t)$.

(iii) Is $x(t)$ continuous on $[0,1]$? Explain.

(iv) State a negation of definiton of uniform convergence, and thus

(v) Show that the given sequence does not converge uniformly.

Hint: Take $\epsilon_o = \frac{1}{2}$ and t_0 such that $t_0^N \geq \frac{1}{2}$ for any N. (2,2,1,3,5)

QUESTION B2 [*20 Marks*]

B2. (a) Let $X = R^2$

(i) Define a Rasberry pickers (or a lift) distance $d(x, y), x, y \in X$.

(ii) Show that (X, d) is M.S. (2,6)

(b) Prove M-test for uniform convergence

(i) Necessary condition

(ii) Sufficient condition (4,4)

(c) Let $X = R^2, A \subset X, B \subset X$,

$A = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$

$B = \{(1, 0)\}$ just a point.

Find the distance $d(A, B)$. Is $d(A, B)$ a metric? Explain (4)

QUESTION B3 [*20 Marks*]

B3. (a) Let $X = R^2$

(i) Define a London (or UK rail) distance $d(x, y), x, y \in X$.

(ii) Show that (X, d) is M.S. (2,5)

(b) (i) Define an open ball in M.S. and sketch in R^2 for

(ii) Euclidean metric

(iii) Chicago metric (3,3,4)

(c) Is a set of irrational numbers closed in R? Give an example to illustrate your statement. (3)

QUESTION B4 [*20 Marks*]

B4. (a) Let $X = R^n$, and $d\infty(x, y) = \max\{x_i = y_i|, i = 1, 2, n\}$, in the usual notations. Show that $(X, d\infty)$ is M.S. (6)

(b) Show that $d\infty$ and Euclidean d metrices are equivalent

Hint: Show $d \geq d\infty \geq \frac{d}{\sqrt{n}}$. (6)

(c) (i) Show that if $f : R^2 \to R$ is continuous and if A is closed in R, then $f^{-1}(A)$ is closed in R^2,

(ii) and thus show that the set D

$D = \{(x, y) : xy \geq 1\}$ is closed in R^2. (5,3,)

QUESTION B5 [*20 Marks*]

B5. (a) Let X be a set of bounded functions, and

$$d(x, y) = sup\,|x(t) - y(t)|, t \in [a, b] .$$

Show that (X, d) is M.S. (6)

(b) Let $x, y : [0, 1] \to R$ be given by $x(t) = \frac{2}{3}t$ and $y(t) = [t]$ (the interger part of t). Evaluate the distance between these functions using "sup" metric. (3)

(c) (i) Define a convergent sequence in M.S. and thus

(ii) Show that the sequence $x_n = \left(1 - \frac{1}{2^{n-1}}, \frac{n^2+1}{2n^2}\right), n = 1, 2...,$

converges to $(1, \frac{1}{2})$ in Euclidean metric, and diverges in discrete metric. (2,5)

(d) Prove that if a set A is complete in M.S. then it is closed. (4)

END OF EXAMINATION PAPER