University of Swaziland

Supplementary Examination, 2017/2018

B.Sc./ B.Ed./ BASS IV

Title of Paper : Metric Spaces

Course Number : M431

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked**.
2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

A1. Let $x, y \in X$, where X is non-empty set.

(a) Define a metric $d(x, y)$ on X. (3)

(b) Show that $d(x, y) \geq 0$. (3)

A2. In a metric space (M.S.) (X, d) given $X = R^2, x = (1, -1), y = (3, 2)$. Find a distance $d(x, y)$ in the following M.S.

(a) Euclidean

(b) Chacago

(c) New York (3,3,3)

A3. Let $X = R^2$ and $d(x, y) = |x_1| + |y_1| + |x_2| + |y_2|$, where $x(x_1, x_2) \in X, y(y_1, y_2) \in X$. Is (X, d) a M.S.? Explain. (4)

A.4 Let the sequence of the continuous on $[a, b]$ functions $x_n(t)$, $t \in [a, b]$, converges, to $x(t)$ with the "max" metric. Prove that $x_n(t_0) \to x(t_0)$ as $n \to \infty$ for any $t_0 \in [a, b]$. (4)

A5. State (without proof) M-test for uniform convergence. (3)

A6. Prove that if $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in M.S., then $x = y$ (4)

A7. Condider M.S. $(X, d), X = R^2$ with the Euclidean metric. Let $A \subset X$. Find a diameter of A, if

(a) $A = (x_1, x_2) : x_1^2 + x_2^2 = 1$,

(b) $A = (x_1, x_2) : x_1^2 + x_2^2 < 1$,

(c) $A = (x_1, x_2) : |x_1| < 2, |x_2| \leq 1$. (2,2,2)

A8. Give definition of a set $A \subset X$ in M.S. (X, d) which is

(a) Complete

(b) Compact. (2,2)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B1 [*20 Marks*]

B1. (a) Let X be a complex plane

(i) prove that if $x, y, \in X$ then

$$|x + y| \leq |x| + |y|$$

(ii) and thus prove that (X, d) is M.S. where $d(x, y) = |x - y|$. (3,4)

(b) Given a sequence $x_n = \frac{nt}{1 + nt}, n = 0, 1 \ldots \ t \in [0, 1]$

(i) Sketch the graphs of $x_n(t)$ for some n.

(ii) Find a limit function $x(t) = \lim x_n(t)$ as $n \to \infty$

(iii) Is $x(t)$ continuous on $[0, 1]$? Explain.

(iv) State a negation of definition of uniform convergence, and thus

(v) Show that the given sequence does not converge uniformly.

Hint: Take $\epsilon_0 = \frac{1}{2}$ and t_0 such that $Nt_0 < 1$ for any N. (2,2,1,3,5)

QUESTION B2 [*20 Marks*]

B2. (a) Let $X = R^2$

(i) Defne a New York (or two skyscrapers) distance $d(x, y), x, y \in X$.

(ii) Show that (X, d) is M.S. (2,6)

(b) Given a sequence

$$x_n(t) = n^2 t(1 - t)^n n = 0, 1, \ldots; t \in [0, 1].$$

(i) Show that the sequence converges pointwise on $[0, 1]$.

(ii) Apply M-test to show that convergence is not uniform on $[0, 1]$.

Hint: $\left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$. (2,5)

(c) Let $X = R^2$. Consider three subsets

$A = (x_1, x_2) : x_1^2 + x_2^2 < 1$,

$B = (1, 0)$,

$C = (-1, 0)$.

Find the distances $d(C, B), d(C, A), d(A, B)$ and thus show that this distance contradicts condition M3 of a metric. (5)

QUESTION B3 [20 Marks]

B3. (a) Let $X = R^2$.

(i) Define a Chicago (or an skyscraper) distance $d(x, y), x, y \in X$.

(ii) Show that (X, d) is M.S. (2,4)

(b) (i) Define an open ball is M.S. and sketch it in R^2 for

(ii) $d\infty = \max\{|x_1 - y_1|, |x_2, -y_2|\}$

(iii) discrete metric (3,3,4)

(c) Consider a set $A \subset R^2$

$$A = (x, y) : x^2 + y^2 < 2 ,$$

and a sequence x_n in A.

$$x_n = \left(1 - \frac{1}{n}, 1 - \frac{1}{n}\right), n = 1, 2, ...,$$

Show that the set A is not closed. (4)

QUESTION B4 [20 Marks]

B4. (a) Let $X = R^n$

(i) Define a Post-mail distance $d(x, y)$ for $x, y \in X$.

(ii) Show that (X, d) is M.S. (2,4)

(b) Given $X = R^2$. Show that the Chicago metric d_1 and the Euclidean metric d_2 are equivalent.

Hint: Show $d_1 \geq d_2 \geq \frac{d_1}{\sqrt{2}}$. (6)

(c) (i) Show that if $f : R^2 \to R$ is continuous and if A is closed in R, then $f^{-1}(A)$ is closed in R^2, and thus

(ii) Show that the set $D, D = \{(x, y) : ax + by \leq c, \quad a, b, c\}$ reals not all zero is closed in R^2 (5,3)

QUESTION B5 [20 Marks]

B5. (a) Let X be the set of continuous functions from $[a, b]$ to R, and for any $x, y, \in X$ define

$$d(x, y) = \max |x(t) - y(t)| : t \in [a, b] .$$

Prove that (X, d) is M.S. [5]

(b) For the metric $d(x, y)$ in (a) find $d(x, y)$ if $x = \sin t, \quad y = \cos t, \quad t \in [0, 2\pi]$ (3)

(c) (i) Define a convergent sequence in M.S. and thus

(ii) Show that the sequence Z_n,

$$Z_n = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}, n = 1, 2, \ldots,$$

converges to 1 in complex plane metric and diverges in discrete metric. (2,5)

(d) Prove that if a set A is compact in M.S., then it is complete (5)

END OF EXAMINATION PAPER