UNIVERSITY OF SWAZILAND

FINAL EXAMINATION, 2017/2018

B.Sc. II, BASS II, BED. II, B.ENG. II

Title of Paper : Linear Algebra

Course Number : MAT221/M220

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:

   a. SECTION A (40 MARKS)
      - Answer ALL questions in Section A.

   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer ANY THREE (3) questions in Section B.
      - If you answer more than three (3) questions in Section B, only the first three questions answered in Section B will be marked.

2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

QUESTION A1

(a) Let V be all ordered pairs of real numbers. Define addition and scalar multiplication as follows:

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$$

and $\propto (x_1, y_1) = (\propto x_1 + \propto - 1, \propto y_1 + \propto - 1)$. Show that V is a vector space. (10 marks)

(b) Given that $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{pmatrix}$. Use the augmented matrix $[A:I]$ to compute A^{-1}. (5 marks)

(c) Determine whether the following has a non-trivial solution:

$$\begin{aligned} 2x + y - z + 2w &= 0 \\ x + y + z + w &= 0 \\ 3x + 2y + 2z + 2w &= 0 \end{aligned}$$

(5 marks)

QUESTION 2

(a) Use Cramer's rule to solve (i) and use Gaussian elimination to solve (ii)

(i)

$$\begin{aligned} 2x + 2y + z &= 1 \\ 3x + y + z &= 2 \\ x + y + z &= 2 \end{aligned}$$

(ii)

$$\begin{aligned} x_1 + z_2 + x_3 &= 3 \\ 2x_1 + 3x_2 + x_3 &= 5 \\ x_1 + x_2 = 2x_3 &= {}^{-}5 \end{aligned}$$

(10 marks)

(b). Find the inverses by inspection

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(5 marks)

(c) Given that $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$. Verify Cayley-Hamilton theorem for the matrix A.

(5 marks)

SECTION B: ANSWER ANY THREE QUESTIONS

QUESTION B3

(a) Determine whether the sets of vectors in the vector space V are linearly dependent or independent.

(i) $\{2x^2 + x, x^2 + 3, x\}$ $\quad V = P_2(x)$

(ii) $\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\right\}$ $\quad V = M_2(R)$

(10 marks)

(b) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of non-zero vectors in a vector space V. Prove that S is linearly dependent $\Leftrightarrow$ one of the vectors is a linear combination of the preceding vectors in S. (10 marks)

QUESTION B4

(a) Find conditions for λ and μ for which the following system has

(i) a unique solution
(ii) no solution or
(iii) infinitely many solutions

$$\begin{aligned} x + y - 4z &= 0 \\ 2x + 3y + z &= 1 \\ 4x + 4y + \lambda z &= \mu \end{aligned}$$

(10 marks)

(b) Let $T : R^2 \to R^3$ be given by $T(x, y) = (x - 2y, 2x + y, x + y)$

(i) Find the standard matrix of T
(ii) Find the matrix of T with respect to B' and B where
$B' = \{(1, -1), (0, 1)\}$ and $B = \{(1, 1, 0), (0, 1, 1), (1, -1, 1)\}$

(10 marks)

QUESTION B5

(a) Prove that if a homogeneous system has more unknowns than the number of equations then it has a non-trivial solution. (10 marks)

(b) Find the characteristic polynomial, eigenvalues and eigenvectors of the following matrix:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

(10 marks)

QUESTION B6

(a) Show that B is a basis for R^3

$$B = \{(0, 2, 1), (1, 0, 2), (1, -1, 0)\}$$ (5 marks)

(b) Show that the vector $\begin{pmatrix} 12 \\ 12 \\ -3 \end{pmatrix}$ is a linear combination of the vectors

$$(2, 0, 1)^T, (4, 2, 0)^T, (1, 3, -1)^T$$ (5 marks)

(c) Let $B_1 = \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ and $B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ be bases. Find the transition matrix from B_1 to B_2 (10 marks)

QUESTION B7

(a) Given that

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & 5 & -3 \\ 0 & -1 & 3 & 0 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Use the augmented matrix $\left[A : I\right]$ to find A^{-1}.

(b) In (a) above find a finite sequence of elementary matrices $E_1, E_2, \dots, E_k$ such that $E_k E_{k-1} \dots E_I A = I$. (13 marks)

(c) Evaluate the following determinant using cofactor expansion along the second row:

$$\begin{vmatrix} 3 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -3 & 2 \end{vmatrix}$$

(4 marks)

(d) Determine whether the system has a non-trivial solution

$$\begin{aligned} -x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \\ 3x_1 + x_2 - x_3 + 2x_4 &= 0 \\ x_1 - 2x_2 + 3x_3 - x_4 &= 0 \end{aligned}$$

(3 marks)