UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2017/2018

B.Sc. II, BASS II, BED. II, B.ENG. II

Title of Paper : Linear Algebra

Course Number : MAT221/M220

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:

   a. SECTION A (40 MARKS)
      - Answer ALL questions in Section A.

   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer ANY THREE (3) questions in Section B.
      - If you answer more than three (3) questions in Section B, only the first three questions answered in Section B will be marked.

2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

QUESTION A1

(a) Let $S = \{v_1, v_2, \dots, v_n\}$ be a set of non-zero vectors in a vector space V. Explain precisely what is meant by each of the following statements:

(i) S spans V (2 marks)
(ii) S is linearly dependent in V (2 marks)
(iii) S is a basis for V (2 marks)

(b) Let $T: R^3 \to R^3$ be a linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - z \\ x + y - z \\ z \end{pmatrix}$

(i) Find the matrix A of T with respect to the standard basis.

(ii) Find the matrix A' of T with respect to the basis

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(iii) Find a 3×3 transition matrix P

(iv) Show that $A' = PAP^{-1}$

(14 marks)

QUESTION A2

(a) Let V be a vector space, A and B be finite sets of non-zero vectors in V such that $A \subset B$. Show that

(i) A linearly dependent $\Rightarrow B$ is also linearly dependent

(ii) B linearly independent $\Rightarrow A$ is also linearly independent

(10 marks)

(b) (i) Express A and A^{-1} as products of elementary matrices where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

(ii) Compute the product for A^{-1} and show that A^{-1} is the inverse of A

(10 marks)

SECTION B: ANSWER ANY THREE QUESTIONS

QUESTION B3

(a) Find the characteristic polynomial, eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$$

(14 marks)

(b) Use Cramer's rule to solve

$$\begin{pmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

(6 marks)

QUESTION B4

(a) Find values of k for which the linear system has

(i) a unique solution
(ii) no solution or
(iii) infinitely many solutions

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -k^2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ k \end{pmatrix}$$

(10 marks)

(b) (i) Find the co-ordinate vector of $\begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}$ with respect to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(6 marks)

(c) Find the standard matrix for the linear transformation $T: R^2 \to R^3$ where

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ x \\ 4x-2y \end{pmatrix}$$

(4 marks)

QUESTION B5

(a) Let V be the set of all ordered pairs of real number. Define addition and scalar multiplication as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$$
$$\propto (x_1, y_1) = (\propto x_1 + \propto - 1, \propto y_1 + \propto - 1).$$

Show that V is a vector space. (10 marks)

(b) Show that the vector $(1, 2, 4)$ is a linear combination of the vectors $(0, 2, 1)$, $(1, 1, 2)$ and $(1, -1, -1)$. (6 marks)

(c) Determine whether the following has a non-trivial solution:

$$x_1 + x_2 - x_3 + x_4 = 0$$
$$2x_1 + x_2 - x_3 + 2x_4 = 0$$
$$3x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

(4marks)

QUESTION B6

(a) Determine whether the following mappings are linear transformations

(i) $T : R^3 \rightarrow R^2$ defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y - z \\ 2x + y \end{pmatrix}$

(ii) $T : R^3 \rightarrow R^3$ defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 1 \\ x + y \\ x + 2y \end{pmatrix}$

(10 marks)

(b) Prove that the set $B = \{x^2 + 1, x - 1, 2x + 2\}$ is a basis for the vector space $V = P_2(x)$ (10 marks)

QUESTION B7

(a) Let $\{v_1, v_2, \ldots, v_n\}$ be a set of non-zero vector in a vector space V. Prove that S is linearly independent if and only if one of the vectors v_1 is a linear combination of the preceding vectors in S. (10 marks)

(b) Verify the Cayley-Hamilton theorem for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

(10 marks)