UNIVERSITY OF SWAZILAND

EXAMINATION, 2017/2018

BASS II, B.Ed (Sec.) II, B.Sc. II

Title of Paper : Foundations of Mathematics

Course Number : MAT231/M231

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, A2, B2, ..., B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*20 Marks*]

((a) Define each of the following.

i. A *proposition*. (2)
ii. A *tautology*. (2)
iii. A *relation* from a set A into a set B. (2)
iv. A *partial order* on a set A. (2)
v. A *function* from a set A into a set B. (2)
vi. A *one-to-one* function $f : A \to B$. (2)
vii. An *onto* function $f : A \to B$. (2)

(b) i. What does it mean to say that the sets A and B have the same cardinality? (3)
ii. Let A be a set. Explain what it means to say A is *finite* and define the cardinality of a finite set. (3)

QUESTION A2 [*20 Marks*]

(a) Write down (i.) the inverse, (ii.) the converse, and (iii.) the contrapositive of the following statement.

"If I'm wearing my lucky hat, then I will pass this exam."

(6)

(b) Let $p(x) : x > -1$ and $q(x) : x \in \{0,1,2\}$ be predicates. Find the truth values of the following propositions.

i. $p(1) \to q(-1)$ ii. $p(1) \wedge \neg p(-1)$ iii. $\neg(p(2) \vee q(2))$. (6)

(c) For $x, y \in \mathbb{R}$, let $p(x,y)$ be the predicate $x < y$. Determine the truth values of the following propositions.

i. $(\forall x)(\exists y)p(x,y)$ ii. $(\exists x)(\forall y)p(x,y)$ (4)

(d) Find $\neg\{(\forall x)(\exists y)[p(x,y) \wedge (\exists y)\neg q(y)]\}$. (4)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [20 *Marks*]

(a) Show that the following compound proposition is a tautology. (4)

$$[(p \to q) \wedge \neg q] \to \neg p.$$

(b) Use a truth table to determine whether or not the following argument is valid. (5)

$$\begin{array}{l} p \vee q \\ p \to \neg q \\ p \to r \\ \therefore\ r \end{array}$$

(c) Use fundamental logical equivalences (not truth tables) to prove the following logical equivalence. (5)

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

(d) In each of the following, state whether the argument is valid or not. If valid, state the rule of inference that guarantees its validity. Otherwise, state whether the inverse or converse error is made.

i. If all students are happy, then there is no class boycott.
There is no class boycott.
Therefore, all students are happy. (2)

ii. If I am well prepared, then I will pass this exam.
I am not well prepared.
Therefore, I will not pass this exam. (2)

iii. If Sipho passed his exams, then Sipho is happy.
Sipho is not happy.
Therefore, Sipho did not pass his exams. (2)

QUESTION B4 [20 *Marks*]

(a) Prove: *If* x^2 *is odd, then* x *is odd.* (5)

(b) Prove: *The square of any integer is of the form* $4k$ *or* $4k+1$ *for some integer* k. (5)

(c) Prove: *The set of prime numbers is infinite.* (5)

(d) Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$, and $b \neq 0$. Prove: *If* $a \mid b$ *and* $b \mid c$*, then* $a \mid c$. (5)

QUESTION B5 [20 *Marks*]

(a) Let A and B be sets in a universal set U. Prove each of the following.

i. $A \cap B \subseteq A$ (3)

ii. $A \subseteq B$ if and only if $A \cap B = A$. (6)

iii. $(A \cup B)^c = A^c \cap B^c$. (6)

(b) i. Define a *partition* of a set A. (2)

ii. Let $A = \{1,2,3,4,5,6\}$, $A_1 = \{1\}, A_2 = \{2,3\}, A_3 = \{4,5,6\}$. Show that $\{A_1, A_2, A_3\}$ is a partition of A. (3)

QUESTION B6 [20 *Marks*]

(a) Let A be a set of nonzero integers and define the relation $\sim$ on $A \times A$ by $(a,b) \sim (c,d)$ whenever $ad = bc$. Show that $\sim$ is an equivalence relation on $A \times A$. (6)

(b) Let $A = \{1,2,3,4,5,6\}$. Given that the relation R on A defined by

$$R = \{(1,1),(1,5),(2,2),(2,3),(2,6),(3,2),(3,3),(3,6),(4,4),(5,1),(5,5),(6,2),(6,3),(6,6)\}$$

is an equivalence relation, find the equivalence classes of R. (4)

(c) Determine whether or not each of the following relations is a function from $A = \{1,2,3\}$ to $B = \{a,b,c\}$. Explain your answer. (2,2

i. $f = \{(1,a),(2,b),(3,b)\}$. ii. $g = \{(1,a),(2,b),(1,c)\}$.

(d) Let $\mathscr{A}$ be a collection of sets. Let R be the relation on $\mathscr{A}$ defined by $(A,B) \in R$ if and only if $A \subseteq B$. Show that $\mathscr{A}$ with this relation is a poset. (6)

QUESTION B7 [20 *Marks*]

(a) Let $f(n) = 2^{2n} - 1$. Use mathematical induction to prove that $f(n)$ is divisible by 3 for all integers $n \geq 1$. (6)

(b) Use strong induction to prove: *Any integer $n > 1$ can be written as a product of prime numbers.* (6)

(b) i. Prove that the composition of two injective functions is also injective. (4)

ii. Prove that the composition of two surjective functions is also surjective. (4)

END OF EXAMINATION PAPER