UNIVERSITY OF SWAZILAND

EXAMINATION, 2017/2018

BSc.III, B.Ed (Sec.) III, BASS III, B.Eng IV

Title of Paper : Numerical Analysis I

Course Number : MAT311/M311

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) How many iterations are required to find an approximation of $\sqrt{3}$ correct to within 10^{-4} by using the bisection method on $f(x) = x^2 - 3$ starting on $[-3, 1]$? [4 Marks]

(b) Show that the Newton's method iteration scheme for finding the root of the function $f(x) = \cos(x) - x = 0$ is

$$x_{n+1} = \frac{x_n \sin(x_n) + \cos(x_n)}{\sin(x_n) + 1}$$

[4 Marks]

(c) Consider the equation $f(x) = 0$ on the interval $[1, 2]$ with $f(x) = e^x - 2x - 1$. Prove that there is a root in $[1, 2]$. [4 Marks]

(d) Convert the 32-bit floating-point number

0 1000 1010 010 0010 0011 1010 0000 0000

to its decimal equivalent. [6 marks]

(e) Convert the decimal number $\frac{43}{5}$ to its binary equivalent. [6 Marks]

(f) Given the following data table

-2	0	2
15	-1	7

derive the Newton interpolation polynomial method *directly*. [6 Marks]

(g) Consider the following linear system of equations in three unknowns x_1, x_2 and x_3

$$\begin{aligned} -2x_1 + 2x_2 + x_3 &= 3, \\ x_1 + 2x_2 - 3x_3 &= 8, \\ x_1 - 2x_2 + 8x_3 &= 9 \end{aligned}$$

Write down the Gauss-Siedel iteration scheme to solve the system. (DO NOT SOLVE!!!) [5 Marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Evaluate the integral

$$\int_0^2 \ln(1+x)\, dx$$

by the Trapezoidal rule with accuracy $\epsilon = 0.05$ [10 Marks]

(b) Determine the quadrature formula of the form

$$\int_a^b f(x)dx \approx A_0 f(x_0) + A_1 f(x_1) + \ldots + A_n f(x_n)$$

where the interval is $[-2, 2]$ and the nodes are $-1, 0$ and 1. [8 Marks]

(c) Use the quadrature formula derived above to approximate the integral

$$\int_{-2}^{2} (x^2 + 3x - 1)dx$$

[2 Marks]

QUESTION B3 [*20 Marks*]

B3 (a) Construct a Newton's forward difference table corresponding to the following data and find a polynomial of least degree that goes through the points.

x	4	5	6	7
$f(x)$	23	37	56	82

[6 Marks]

(b) Given the following data table

-2	5	6
-4	52	88

Show that the interpolating polynomial derived using the *direct* Newton interpolation approach is

$$P_2(x) = -4 + 8(x+2) + \frac{7}{2}(x-5)(x+2)$$

[7 Marks]

(c) Find the interpolating polynomial passing through the three points

$$(-2, 36), (2, 24), (5, 183)$$

using the Vandermonde matrix. [7 Marks]

QUESTION B4 [*20 Marks*]

B4 (a) Consider the matrix A below,

$$\begin{bmatrix} 4 & -6 & -5 \\ 4 & -5 & -5 \\ -8 & 2 & -6 \end{bmatrix}.$$

Use the naive Gaussian elimination method to factor the matrix in the form $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix. [10 Marks]

(b) Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$$

Suppose that the matrix system is to be solved using the Jacobi method with initial guess $\mathbf{x}^{(0)} = (0,0,0)^T$.

i. Will the Jacobi method converge for this problem? Justify your answer. [2 Marks]

ii. Write down the Jacobi iterative method for solving the matrix system $A\mathbf{x} = \mathbf{b}$ [2 Marks]

iii. Perform (only) two (2) full iterations using the iterative scheme in (ii) to calculate the approximate solution. [6 Marks]

QUESTION B5 [*20 Marks*]

B5 (a) Use Taylor series expansions to derive the following difference formulas

$$f'(x) = \frac{11f(x) - 2f(x-3h) + 9f(x-2h) - 18f(x-h)}{6h} + \frac{h^3}{4}f^{(4)}(\xi)$$

. [8 Marks]

(b) Determine the error term for the formula

$$f'(x) \approx \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)]$$

[6 Marks]

(c) Use the formula in (b) to approximate $f'(1.8)$ with $f(x) = ln(x)$ using $h = 0.1, 0.01, 0.001$. Compute the error in each case. [6 Marks]

QUESTION B6 [*20 Marks*]

B6 (a) Find the roots of the following quadratic equation (as accurately as possible) using eight digits and rounding

$$x^2 - 100000x + 1 = 0$$

[4 Marks]

(b) Given the function $f(h) = \sqrt{9-h} - 3$

(i) find a suitable function $g(h)$ that has been reformulated to be algebraically equivalent to $f(h)$ with the aim of avoiding loss of significance error. [2 marks]

(ii) Compare the results of calculating $f(0.0001)$ and $g(0.0001)$ using **six digits** and **chopping**. [4 marks]

(c) State the fixed point theorem. [4 Marks]

(d) Consider the iterative scheme

$$x_{k+1} = (\alpha + 1)x_k - x_k^2$$

i. Find the fixed points of the scheme. [2 Marks]

ii. Show that the interval where this scheme is guaranteed to converge is

$$1 + \frac{\alpha}{2} < x < \frac{\alpha}{2}$$

[4 Marks]

END OF EXAMINATION PAPER