UNIVERSITY OF SWAZILAND

FINAL EXAMINATION, 2017/2018

B.Sc. III, B.Ed III, B.Eng III, BASS III

Title of Paper : Vector Analysis

Course Number : M312/MAT312

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
    a. SECTION A (40 MARKS)
        - Answer **ALL** questions in Section A.
    b. SECTION B
        - There are FIVE (5) questions in Section B.
        - Each question in Section B is worth 20 Marks.
        - Answer **ANY THREE (3)** questions in Section B.
        - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked**.
2. Show all your working.

Special Requirements: None

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: Answer ALL Questions

A1. Find the equation of the plane which passes through the point $(2, 1, -3)$ and is normal to the vector $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$. (5)

A2. Express the vector $\overline{F} = (-y, x, 0)$ in spherical coordinates. (6)

A3. Given the paraloloid of revolution $x^2 + y^2 - z = 1$. Find the unit normal at the point $(1, 1, 1)$. (5)

A4. Show that the vector $\overline{F} = 3yz^2\hat{\boldsymbol{i}} + 4x^3z^2\hat{\boldsymbol{j}} - 3x^2y^2\hat{\boldsymbol{k}}$ is solenoidal. (4)

A5. Find the total work done in moving a particle in a force field $\overline{F} = 3xy\hat{\boldsymbol{i}} - 5z\hat{\boldsymbol{j}} + 10x\hat{\boldsymbol{k}}$ along a curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$. (5)

A6. (a) Apply the Green's theorem to show that the area bounded by a simple closed curve c is given by $\frac{1}{2}\oint xdy - ydx$. (b) Use line integral to find the area enclosed by the curves $y = x$ and $y^2 = 4x$ (5) (5)

A7. Verify divergence theorem for a sphere of radius R with center at the origin and with

$$\overline{A} = xi + yj + 2k, \quad |\overline{A}| = R$$

Hint: Volume of sphere $\frac{4}{3}\pi R^2$, surface area $4\pi R^2$. (5)

SECTION B: Answer Any *THREE* Questions

QUESTION B1 [*20 Marks*]

B1. (a) Find the equation of the plane which contains the lines

$$\frac{x-1}{-1} = \frac{y-2}{2} = \frac{z-3}{4} \text{ and } \frac{x+1}{2} = \frac{y-4}{-2} = \frac{z-2}{1}.$$

(6)

(b) Find the acute angle between the planes $2x - y + z + 8 = 0$ and $3x + 2y - z = 0$ (5)

(c)

(i) Find the directional derivative of $f = xyz^2$ at point $(1, 0, 3)$ in the direction of the vector $\overline{a} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

(ii) Compute the greatest rate of change of f.

(iii) The direction of the maximum rate of increease of f. (5,2,2)

QUESTION B2 [*20 Marks*]

B2. (a) Consider the toroidal curvilinear coordinates R, θ, φ which are defined in terms of rectangular cartesian coordinates x, y, z by

$$x = (a - R\cos\theta)\cos\varphi, \quad y = (a - R\cos\theta)\sin\varphi, \quad z = R\sin\theta$$

where $a =$ constant and $R < a$.

(i) Show that it is an orthogonal system of coordinates.

(ii) Find the Lame parameters. (6,4)

(b) Given an arbitrary system of curvilinear orthogonal coordinates q_1, q_2, q_3. Derive the formulas for

(i) Velocity,

(ii) Acceleration. (3,7)

QUESTION B3 [*20 Marks*]

B3. (a) Prove the following formulae

(i) $\nabla \times (f\overline{A}) = f\nabla \times \overline{A} + \nabla f \times \overline{A}$.

(ii) $\nabla \times \overline{r} = 0$

(iii) $(\overline{A} \cdot \nabla)\overline{r} = \overline{A}$ (4,2,3)

(b) Consider a vector field

$$\overline{F} = (y^2 + 2xz^2 - 1)\hat{\imath} + 2xy\hat{\jmath} + 2x^2z\hat{\boldsymbol{k}}.$$

(i) Is $\overline{F}$ irrotational? Explain

(ii) Is $\overline{F}$ conservative? Explain

(iii) If so, find a scalar potential (4,2,5)

QUESTION B4 [*20 Marks*]

B4. (a) If $\overline{U} = (x, 2y, 3z)$, evaluate

(i) $\displaystyle\int_0^A \overline{U} \cdot d\overline{r}$,

(ii) $\displaystyle\int_0^A \overline{U} \times d\overline{r}$,

(iii) $\displaystyle\int_0^A \overline{U} ds$

along the curve $\overline{r} = \left(t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{3}\right)$ from the origin to the point $A(1, \frac{1}{\sqrt{2}}, \frac{1}{3})$. (5,3,4)

(b) Verify Stoke's theorem for

$$\overline{A} = (2x - y)\hat{\boldsymbol{i}} - yz^2\hat{\boldsymbol{j}} - y^2z\hat{\boldsymbol{k}},$$

where S is half surface of the sphere $x^2 + y^2 + z^2 = 1$, and C is its boundary. (8)

QUESTION B5 [*20 Marks*]

B5. Apply the divergence theorem

(a) To derive theorem of the gradient

$$\int\int_V\int \nabla\varphi dV = \int\int_S \varphi d\overline{S}.$$

(6)

(b) To derive Green's first identity

$$\int\int_V\int (u\nabla^2 w + \nabla u \cdot \nabla w)dV = \int\int_S (u\nabla w) \cdot d\overline{S}.$$

(6)

(c) To transform the following double integral to a triple integral for the specified closed surface S.

$$\int\int_S (x^3dydz + x^2ydzdx + x^2zdxdy),$$

S is a surface of a cylinder $x^2 + y^2 = 4, 0 \leq z \leq 2$, and the circular disks $z = 0$ and $z = 2 \ \ (x^2 + y^2 \leq 4)$. (8)

END OF EXAMINATION PAPER