UNIVERSITY OF SWAZILAND

RESIT/SUPPLEMENTARY EXAMINATION, 2017/2018

**BASS III, B.Ed (Sec.) III, B.Sc. III, B.Eng. III**

**Title of Paper** : Complex Analysis

**Course Number** : MAT313/M313

**Time Allowed** : Three (3) Hours

**Instructions**

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

**Special Requirements: NONE**

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

## SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

### QUESTION A1 [*40 Marks*]

a) Compare and contrast the properties of the functions $e^x$ and $e^z$ where $x \in \mathbb{R}$ and $z \in \mathbb{C}$. [5]

b) Express $(i-1)^{1+i}$ in the form $a+ib$. [5]

c) Express $\sin^{-1}(1)$ in the form $a+ib$. [5]

d) Let $C$ be a positively oriented boundary of the square whose sides lie along the lines $x = \pm 3$ and $y = \pm 3$. Evaluate

$$\int_C \frac{e^z}{z-i} dz.$$

[5]

e) Let $C$ be a positively oriented circle such that $|z| = 4$. Evaluate

$$\int_C \frac{z-9}{(z+3i)(z-3i)} dz$$

[5]

f) Find the Laurent series that represents the function

$$f(z) = \frac{\pi}{z(z-i)(z+i)}$$

in the domain $0 < |z| < 1$. [5]

g) Suppose that

$$g(z) = \alpha(x,y) + i\beta(x,y)$$

and it's conjugate are both analytic in a given domain $D$. Show that $g(z)$ must be constant throughout $D$. [7]

h) Show that $\displaystyle\int_C \frac{dz}{z-i} = 2\pi i$ where $C$ is the circle $|z-i| = 4e$. [3]

## SECTION B: ANSWER ANY *THREE* QUESTIONS

### QUESTION B2 [*20 Marks*]

a) Determine if the function

$$g(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$$

is analytic everywhere or not? If $g(z)$ is analytic, find $g'(z)$. [10]

b) Determine whether $u(x, y) = 4xy^3 - 4x^3y + x$, is harmonic. If $u$ is harmonic, find the harmonic conjugate $v$ and the analytic function $f(z) = u + iv$ with $f(1 + i) = 5 + 4i$. [10]

### QUESTION B3 [*20 Marks*]

a) Show that

$$\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

[10]

b) Solve for $z$ and express $z$ in the form $a + ib$

i) $e^{2z} = 1 + i\sqrt{3}$ [5]

ii) $\ln\left(\frac{i-z}{1+z}\right) = (1 - \pi i)$ [5]

### QUESTION B4 [*20 Marks*]

a) Evaluate $\displaystyle\int_C \frac{4z^5}{(z-3)^3}dz$ if $C$ is

i) the circle $|z + 3| = 9$ [7]

ii) the circle $|z - i| = 1$ [3]

b) Prove that if a function $f(z) = \phi(x, y) + i\eta(x, y)$ is analytic in a domain $D$, then $\phi(x, y)$ and $\eta(x, y)$ are harmonic in $D$. [10]

**QUESTION B5** [*20 Marks*]

a) Find the Laurent series that represents the function

$$f(z) = \frac{2}{z(z-1)}$$

in the domain $0 < |z| < 1$. [8]

b) Suppose that

$$g(z) = \alpha(x, y) + i\beta(x, y)$$

and it's conjugate are both analytic in a given domain $D$. Show that $g(z)$ must be constant throughout $D$. [12]

**QUESTION B6** [*20 Marks*]

a) Evaluate $\int_C \frac{\sin(\pi z) + \cos(\pi z)}{(z-2)(z-1)} dz$ if $C$ is a positively oriented circle such that $|z| = 3$. [8]

b) Let $C$ be a positively oriented circle such that $|z| = 2$. Evaluate

$$\int_C \frac{\cosh(\pi z) dz}{z^2 + 1}$$

[6]

c) Let $C$ be a positively oriented circle such that $|z| = 4$. Using Cauchy's residue theorem, evaluate

$$\int_C \frac{z(z-2)dz}{(z+1)^2(z^2+4)}$$

[6]

END OF EXAMINATION PAPER