UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2017/2018

B.Sc. III, BASS III, B.Ed. III

Title of Paper : Abstract Algebra I

Course Number : M$323/MAT324

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:

   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.

   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, only **the first three questions answered in Section B will be marked.**

2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION A1

(a) Prove that if $(a, s) = 1$ and $(b, s) = 1$ then $(ab, s) = 1 \; \forall \; a, b, s \in Z$ (8 marks)

(b) Give a single numerical example to disprove the following:

"$If \; ax \equiv bx \pmod{n} \; then \; a \equiv b \pmod{n} \; \forall a_1 b_1 n \in Z$" (4 marks)

(c) Prove that every subgroup of a cyclic group is cyclic. (10 marks)

QUESTION A2

(a) Solve the following system:

$$2x \equiv 1 \pmod{5}$$
$$3x \equiv 4 \pmod{7}$$

(7 marks)

(b) Find the number of generators for cyclic groups of order 8 and 60. (7 marks)

(c) Prove that a non-abelian group of order $2p$, p prime contains at least one element of order p. (6 marks)

SECTION B

ANSWER ANY THREE QUESTIONS

QUESTION B3

(a) Prove that every cyclic group is abelian. (5 marks)

(b) Let n be a positive integer greater than 1, and let, for $a, b \in Z$

$$aRb \quad \Leftrightarrow \quad a \equiv b(\bmod n)$$

Prove that R is an equivalence relation on Z (7 marks)

(c) Show that a group G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ (6 marks)

QUESTION B4

(a) Let $\alpha = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 5\,4\,1\,3\,2\,8\,7\,6 \end{pmatrix}$ $\beta = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 8\,5\,6\,4\,1\,7\,2\,3 \end{pmatrix}$

(i) Express α and β as products of disjoint cycles, and then as products of transpositions. For each of them, say whether it is an even permutation or an odd one. (8 marks)

(ii) Compute a^{-1}, $\beta^{-1}\alpha$, $(\alpha\beta)^{-1}$ (6 marks)

(b) Find the greatest common divisor d of the numbers 616 and 427 and express it in the form

$$d = 616x + 427y$$

for some $x, y \in Z$ (6 marks)

QUESTION B5

(a) Compute [do not list] the number of elements in each of the cyclic subgroups

(i) $\langle 30 \rangle \; of \; Z_{42}$

(ii) $\langle 15 \rangle \; of \; Z_{48}$

(6 marks)

(b) For Z_{12}, find all the subgroups and give a lattice diagram (7 marks)

(c) (i) Find all cosets of $H = \{0, 6, 12\}$ in Z_{18}

(ii) Show that the groups Z_6 and S_3 are not isomorphic (7 marks)

QUESTION B6

(a) Prove that every finite group of prime order is cyclic. (5 marks)

(b) Show that the set $G = Q - \{0\}$ with respect to the operation

$$a \times b = \frac{ab}{s} \qquad \forall \quad a, b \in G \text{ is a group.}$$ (9 marks)

(c) Prove that if $a^2 = e \qquad \forall \quad a \in G$, where G is a group then G is abelian. (6 marks)

QUESTION B7

(a) If $\varphi: G \rightarrow H$ is an isomorphism of groups and e is the identity of G then

(i) $(e)\varphi$ is the identity element in H.

(ii) $(a^n)\varphi = [(a)\varphi]^n \quad \forall \; n \in Z^+$ (12 marks)

(b) (i) State Langrange's theorem

(ii) Using (b)(i) above or otherwise, show that Zp has no proper subgroup if p is a prime number. (8 marks)