University of Swaziland

Final Examination, December 2017

B.Sc III, B.A.S.S III, B.Ed III

Title of Paper : Real Analysis

Course Code : MAT331/M331

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. SECTION A(COMPULSORY): 40 MARKS
      Answer ALL QUESTIONS.
   b. SECTION B: 60 MARKS
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

Question 1

(a) Consider a set $A \subseteq \Re$.

(i) When do we say that a set A is bounded? [3]

(ii) Give example(s) of sets which are (I) bounded below not above and (II) bounded neither above nor below. [3]

(iii) Define supremum and infimum of a set A. [3]

(iv) Let S be a set. Prove that supremum of S if it exists, is unique. [4]

(b) (i) Give the ϵ, N definition for the convergence of a sequence $< a_n >$ to a number L. [3]

(ii) Using ϵ, N definition show that $\lim\limits_{n\to\infty} \frac{2n+1}{n+3} = 2$. [5]

(iii) Define limit point and limit of a sequence.
Limit point of a sequence is different from limit of a sequence. Do you agree with this statement? Support your answer with an example. [4]

(c) True or false? If $\lim\limits_{n\to\infty} a_n = 0$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent. Explain your answer. [3]

(d) Let $f : [a, b] \to \Re$ be a function and let $c \in (a, b)$.

(i) True or false? If f is continuous at c, then f is differentiable at c. Explain your answer. [3]

(ii) When do we say that the function f is uniformly continuous? Explain your answer.
Explain the major distinction between continuity and uniform continuity.

[3]

(ii) Define the left-hand derivative of of f at c. [3]

(e) State the Riemann's integrability criterion. [3]

SECTION B: ANSWER ANY 3 QUESTIONS

Question 2

(a) Show that for any $x, y \in \Re$, $\quad |x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$. [6]

(b) If x is a limit point of a set A and $A \subseteq B$, then x is also a limit point of B. [6]

(c) By finding the left-hand and right-hand derivatives of

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

determine $f'(0)$. [8]

Question 3

(a) Find the limit superior and limit inferior of the sequence

$$< (-1)^n (2^n + 3^n) > .$$ [6]

(b) If $a_n = 1 + \frac{(-1)^n}{2n}$, find the least positive integer m such that $|a_n - 1| < \frac{1}{10^3} \quad \forall n > m$. [6]

(c) Prove, by definition, that the sequences whose terms are given by $\frac{n}{n+1}$ is a Cauchy sequence. [8]

Question 4

(a) True or false? If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then it converges absolutely. Explain your answer. [7]

(b) State the conditions for the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots \qquad (a_n > 0 \ \ \forall n)$$

to be convergent. [3]

(c) Let $\sum_{n=1}^{\infty} a_n$ be series of positive terms. Name at least three tests for convergence of this series. [3]

(d) Prove that a series of positive terms ether converges or diverges to ∞. [7]

Question 5

(a) Let $f(x) = \dfrac{x^2+2}{x^2+1}$, then given $\epsilon > 0$, find a real number δ such that $|f(x) - 2| < \epsilon$ whenever $0 < |x| < \delta$. [6]

(b) Prove that the limit of a function at a point, when it exists, is unique. [6]

(c) Show that the function defined by $f(x) = x^2$ is uniformly continuous on $[-2, 2]$.

[8]

Question 6

(a) Let $f(x) = x$ for $x \in [0,1]$ and let $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ be a partition of $[0,1]$. Compute $U(P,f)$ and $L(P,f)$. [8]

(b) Let $f : [-1,1] \to \Re$ be given by

$$f(x) = |x| = \begin{cases} -x & \text{when } x \leq 0 \\ x & \text{when } x > 0 \end{cases}$$

Show that f is Riemann integrable and find $\int_{-1}^{1} f(x)dx$. [12]

End of Examination Paper