University of Swaziland

Supplementary Examination, July 2018

B.Sc III, B.A.S.S III, B.Ed III

Title of Paper : Real Analysis

Course Code : MAT331/M331

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
   Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
   Answer ANY THREE questions.
   Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

Question 1

(a) Consider a set $A \subseteq \Re$.

(i.) Define a limit point of the set A. [3]

(ii.) Explain what is means to say that the set A is bounded below and define inf(A). [3]

(iii.) True or false. The set N of natural numbers is not a neighborhood of any of its points. Explain your answer. [3]

(b) (i) Precisely explain each of the following statements about a sequence $< a_n >$ in $\Re$

(I.) $< a_n >$ is convergent. [3]

(II.) $< a_n >$ is a Cauchy sequence. [3]

(ii.) Let a_n and b_n be two convergent sequences. prove that the sequence $a_n + b_n$ also convergent. [4]

(iii.) Prove that if the sequence $< a_n >$ is convergent, then $< a_n >$ is bounded.

[4]

(c) Let $\sum u_n$ be a series in $\Re$. Precisely explain the following statements.

(i.) $\sum u_n$ converges. [2]

(ii.) $\sum u_n$ is absolutely convergent. [2]

(d) Let $f : [a, b] \to \Re$ be a function and let $c \in (a, b)$. Precisely explain the following:

(i.) f is continuous at c. [3]

(ii.) f is uniformly continuous on $[a, b]$. [3]

(iii.) Prove: If f is differentiable at c then, then f is continuous at c. [4]

(e) State the Riemann's integrability criterion. [3]

SECTION B: ANSWER ANY 3 QUESTIONS

Question 2

(a) Show that for any $x, y \in \Re$, $\quad |x+y| \leq |x| + |y|$. [6]

(b) Show that for a bounded set $S \subseteq \Re$, there exists a positive number A such that $|x| \leq A \;\; \forall x \in S$. Prove that the converse is also true. [6]

(c) Show that

$$f(x) = \begin{cases} x^2 - 1 & \text{when } x \geq 1 \\ 1 - x & \text{when } x < 1 \end{cases}$$

is not differentiable at $x = 1$. [8]

Question 3

(a) Find the limit superior and limit inferior of the sequence

$$< (-10)^n \left(1 + \frac{1}{n}\right) > .$$ [6]

(b) Prove that a sequence $< \dfrac{2n-7}{3n+2} >$

(i) is monotonically increasing, [3]

(ii) is bounded, [3]

(iii) tends to the limit $\frac{2}{3}$. [3]

(c) Prove: If a sequence $< a_n >$ converges l, then every subsequence of $< a_n >$ also converges to l. [5]

Question 4

(a) State the Cauchy convergence criterion for series. [3]

(b) Prove that $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. [5]

(c) Is the converse of the previous statement, (b), true? Justify your answer. [3]

(d) Prove: The necessary and sufficient condition condition for the convergence of a positive term series $\sum a_n$ is that the sequence $< S_n >$ of its partial sums is bounded above. [9]

Question 5

(a) If $a_n = 2 + \frac{(-1)^n}{n^2}$, find the least positive integer m such that $|a_n - 2| < \frac{1}{10^4} \quad \forall n > m$. [8]

(b) Let $f : [a, b] \to \Re$ be a function, and let $c \in (a, b)$. Precisely explain each of the following statements.

(i) A real number p is the left derivative of f at c. [2]

(ii) A real number q is the right derivative of f at c. [2]

(c) Show that the function defined by $f(x) = x^3$ is uniformly continuous on $[-1, 1]$. [8]

Question 6

(a) Let $f(x) = x^2$ for $x \in [0, 1]$ and let $P = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ be a partition of $[0, 1]$. Compute $U(P, f)$ and $L(P, f)$. [10]

(b) From the definition of the Riemann integral show that $\int_1^2 (2x + 3) = 6$. [10]

End of Examination Paper