UNIVERSITY OF SWAZILAND

MAIN EXAMINATION, 2017/2018

BSc.III

Title of Paper : Mathematical Statistics I

Course Number : MAT340

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Let C and D be two events with $P(D) = 0.45$, $P(C) = 0.25$, and $P(C \cap D) = 0.1$. Compute $P(C^c \cap D)$.

[3 Marks]

(b) Let A and B be two events with $P(B) > 0$.

(i) Write down an expression for the conditional probability of A given B.

[3 Marks]

(ii) Determine the conditional probability of A given B in the following cases: A and B are independent; A and B are mutually exclusive.

[2 Marks]

(c) A diagnostic test for a disease gives a positive result with probability 0.98 for people who have the disease, and a negative result with probability 0.99 for people who do not have the disease. Suppose 3% of the population have the disease.

(i) A person is selected at random from the population and given the test. If the result is positive, what is the probability that this person has the disease?

[4 Marks]

(ii) Suppose a person, initially selected at random from the population, is given the test once and the result is positive. This person is then given the test, independently, a second time and the result is again positive. What is the probability that this person has the disease?

[3 Marks]

(d) Find the moment-generating function $M_X(t)$ for a Poisson distributed random variable with mean λ. Use $M_X(t)$ to find $E(X)$ and $Var(X)$.

[10 Marks]

(e) Experience has shown that 30% of all persons afflicted by a certain illness recover. A drug company has developed a new medication. Ten people with the illness were selected at random and received the medication; nine recovered shortly thereafter. Suppose that the medication was absolutely worthless. What is the probability that at least nine of ten receiving the medication will recover?

[6 Marks]

(f) The continuous random variables X and Y have joint probability density function

$$f(x,y) = \begin{cases} 60x^2y, & 0 < x < 1, 0 < y < 1, 0 < x+y < 1, \\ 0, & \text{otherwise} \end{cases}$$

Derive the marginal probability density functions of X and Y and deduce their expectations. Compute and interpret their covariance.

[10 Marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) The discrete random variable U is equally likely to take any of the integer values $1, 2, \ldots, k \quad (k \geq 1)$. Show that

$$E(U) = \frac{k+1}{2} \text{ and } Var(U) = \frac{k^2 - 1}{12}.$$

[You may use the result without proof that $1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$.]

[6 Marks]

(b) The random variable X has the Normal distribution with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), \quad -\infty < x < \infty$$

(i) Show that X has moment generating function $M_X(t) = \exp(\mu t + \frac{\sigma}{2}t^2)$.

[8 Marks]

(ii) For constants a and b, show that the moment generating function of $Y = aX + b$ is

$$e^{bt}M_X(at).$$

Use this result to obtain the moment generating function of

$$Z = \frac{X - \mu}{\sigma}.$$

Deduce the distribution of Z.

[6 Marks]

QUESTION B3 [*20 Marks*]

B3 (a) A soft-drink machine has a random amount Y_2 in supply at the beginning of a given day and dispenses a random amount Y_1 during the day (with measurements in gallons). It is not resupplied during the day, and hence $Y_1 \leq Y_2$. It has been observed that Y_1 and Y_2 have a joint density given by

$$f(y_1, y_2) = \begin{cases} 1/2, & \text{if } 0 \leq y_1 \leq y_2 \leq 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the conditional probability density of Y_1 given $Y_2 = y_2$. Evaluate the probability that less than 1/2 gallon will be sold, given that the machine contains 2 gallons at the start of the day.

[8 Marks]

(b) Let random variables Y_1 and Y_2 have the joint probability density function

$$f(y_1, y_2) = \begin{cases} 2(1 - y_1), & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

and $U_1 = Y_1$ and $U_2 = Y_1 Y_2$. Find the probability density function for U_2.

[12 Marks]

QUESTION B4 [*20 Marks*]

B4 (a) Find the moment-generating function for a gamma-distributed random variable.

[8 Marks]

(b) Let $Y_1, Y_2, \ldots, Y_n$ denote independent random variables with cumulative distribution function $F(y)$ and probability density function $f(y)$.

(i) Derive the probability density function of $Y_{(n)} = max\{Y_1, Y_2, \ldots, Y_n\}$.

[6 Marks]

(ii) Electronic components of a certain type have a length of life Y, with probability density given by

$$f(y) = \begin{cases} (1/100)e^{-y/100}, & \text{if } y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

(Length of life is measured in hours.) Suppose that two such components operate independently and in parallel in a certain system (hence, the system does not fail until both components fail). Find the density function for X, the length of life of the system. Hence compute the probability that $X > 200$ hours.

[6 Marks]

QUESTION B5 [*20 Marks*]

B5 (a) Let Y be a binomial random variable based on n trials and success probability p. Let $E(Y) = np$, show that $\sigma^2 = np(1-p)$.

[10 Marks]

(b) Consider the experiment of tossing a fair coin 3 times. Let X be the number of heads on the first toss and F the number of heads on the first two tosses. Fill the joint probability table for X and F. Compute Cov(X, F).

[10 Marks]

QUESTION B6 [*20 Marks*]

B6 (b) If Y is a beta-distributed random variable with parameters $\alpha > 0$ and $\beta > 0$, its probability density function is

$$f(y) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 < y < 1, \\ 0, & \text{otherwise} \end{cases}$$

Show that $\mu = E(Y) = \frac{\alpha}{\alpha+\beta}$.

[5 Marks]

(c) A gasoline wholesale distributor has bulk storage tanks that hold fixed supplies and are filled every Monday. Of interest to the wholesaler is the proportion of this supply that is sold during the week. Over many weeks of observation, the distributor found that this proportion could be modeled by a beta distribution with $\alpha = 4$ and $\beta = 2$. Find the probability that the wholesaler will sell at least 90% of her stock in a given week.

[5 Marks]

(a) Let X be any continuous random variable, and let $F(x)$ denote its cumulative distribution function. Suppose that U is a continuous random variable with the uniform distribution on the interval 0 to 1, and define the new random variable Y by

$$Y = F^{-1}(U),$$

where $F^{-1}(.)$ is the inverse function of $F(.)$.

(i) By considering the cumulative distribution function of Y, show that Y has the same distribution as X.

[6 Marks]

(ii) Briefly describe a method of simulating pseudo-random variates from a continuous probability distribution, based on this result.

[4 Marks]

END OF EXAMINATION PAPER