University of Swaziland

Final Examination – December 2017

BSc IV, BEng III, BEd IV, BASS IV

Title of Paper	:	Partial Differential Equations
Course Number	:	M415/MAT416
Time Allowed	:	Three (3) hours

Instructions:

1. This paper consists of 2 sections.
2. Answer ALL questions in Section A.
3. Answer ANY 3 (out of 5) questions in Section B.
4. Show all your working.
5. Begin each question on a new page.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A

Answer ALL Questions in this section

A.1 a. Classify each of the following PDEs according to order, linearity and homogeneity.

i. $\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(x\dfrac{\partial u}{\partial x}\right) = u$ [3 marks]

ii. $\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(u\dfrac{\partial^2 u}{\partial x^2}\right) = x$ [3 marks]

b. Find the PDE whose general solution is

$$u(x, y) = \sqrt{x^2 + y^2} + xF(y) + G(y),$$

where F and G are arbitrary functions. [5 marks]

c. Find the eigenvalues and eigensolutions of the Sturm-Liouville system

$$X'' + \alpha^2 X = 0, \ \ X'(0) = X(\ell) = 0,$$

where α and $\ell > 0$ are real constants. [5 marks]

d. Consider the initial-value problem

$$u_t - u_x = 2xu, \ \ u(x, 0) = 4e^{x-x^2}.$$

Solve the problem using

i. the method of characteristics [7 marks]

ii. the method of separation of variables [7 marks]

e. Use *Laplace transforms* to solve the initial-value problem

$$u_t + u = e^{-t}, \ \ u(x, 0) = e^{-x}, \quad x > 0, \ t > 0.$$ [5 marks]

f. The ODE

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) - s(s+1)R = 0, \tag{1}$$

where s is a real constant, arises in the solution of PDEs in spherical domains. Find the general solution of (1). [5 marks]

Section B

Answer ANY 3 Questions in this section

B.1 a. Consider the PDE

$$4u_{xx} + 5u_{xy} + u_{yy} + x = y. \qquad (2)$$

i. Classify (2) as parabolic, hyperbolic or elliptic. [2 marks]

ii. Reduce (2) into its canonical form [9 marks]

iii. Find the general solution of (2). [3 marks]

b. Find the solution of the Cauchy problem of the wave equation [6 marks]

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, & -\infty < x < \infty,\ t \geqslant 0 \\ u(x,0) &= \sin x, & -\infty < x < \infty \\ u_t(x,0) &= 2x, & -\infty < x < \infty. \end{aligned}$$

B.2 a. Consider the function $f(x) = x,\ \ 0 < x < 2$.

i. Make a sketch of the even extension of $f(x)$ over the interval $-4 < x < 8$. [3 marks]

ii. Find the half-range cosine expasion of $f(x)$. [6 marks]

iii. Use the results from ii. and Parseval's identity to find the value of the infinite sum

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots.$$ [5 marks]

b. Use the generating function

$$G(z,t) = \exp\left[\frac{t}{2}\left(z - \frac{1}{z}\right)\right] = \sum_{n=-\infty}^{\infty} z^n J_n(t)$$

to derive the recurrence relation

$$\frac{2n}{t} J_n(t) = J_{n-1}(t) + J_{n+1}(t)$$

for Bessel functions. [6 marks]

B.3 Consider a semi-infinite plate of width π whose two sides are insulated while the base is kept at x. The steady-state temperature distribution of the plate is governed by the system

$$\begin{aligned} \nabla^2 u = u_{xx} + u_{yy} &= 0, & 0 \leq x \leq \pi, y \geqslant 0 \\ u(x,0) &= x, & 0 \leq x \leq \pi \\ u_x(0,y) = u_x(\pi,y) &= 0, & 0 < y < \infty. \end{aligned}$$

Find the function $u(x,y)$ for the equilibrium temperature distribution.

[20 marks]

B.4 a. Find the PDE satisfied by u, defined by

$$u = xy + F(x^2 + y^2 - u),$$

where $u = u(x,y)$ and F is an arbitrary function. [10 marks]

b. Find the general solution of the PDE

$$(x - y)u_x + yu_y = xu.$$

[10 marks]

B.5 Consider the non-homogeneous boundary-value problem

$$\begin{aligned} u_t - u_{xx} &= e^{-t}\sin\left(\tfrac{1}{2}x\right), & 0 < x < \pi,\ t > 0 \\ u(0,t) = u_x(\pi,t) &= 0, & t \geqslant 0 \\ u(x,0) &= 4\sin\left(\tfrac{3}{2}x\right), & 0 \leqslant x \leqslant \pi \end{aligned}$$

a. Show that the eigenfunctions and eigenvalues of the associated homogeneous problem are given by $\sin \alpha_n x$ and $\alpha_n = \frac{1}{2}(2n-1)$, $n = 1, 2, 3, \cdots$, respectively. [7 marks]

b. Hence, or otherwise, solve the non-homogeneous problem. [13 marks]

END OF EXAMINATION