University of Swaziland

Supplementary/Re-sit Examination – July 2018

BSc IV, BEng III, BEd IV, BASS IV

Title of Paper	:	Partial Differential Equations
Course Number	:	M415/MAT416
Time Allowed	:	Three (3) hours

Instructions:

1. This paper consists of 2 sections.
2. Answer ALL questions in Section A.
3. Answer ANY 3 (out of 5) questions in Section B.
4. Show all your working.
5. Begin each question on a new page.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A

Answer ALL Questions in this section

A.1 a. Classify each of the following PDEs according to order, linearity and homogeneity.

i. $x^2 u_{xx} - y^2 u_y = e^{-x^2}$ [3 marks]

ii. $u_x + (x-u)u_y = x^2 u$ [3 marks]

b. Find the PDE whose general solution is

$$u(x,y) = \frac{1}{x + F(y)},$$

where F is an arbitrary function. [5 marks]

c. Evaluate

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, \mathrm{d}x,$$

where both m and m are integers. [5 marks]

d. Consider the initial-value problem

$$xu_x + u_y = -u, \quad u(x,0) = 4x.$$

Solve the problem using

i. the method of characteristics [7 marks]

ii. the method of separation of variables [7 marks]

e. Use *Laplace transforms* to solve the initial-value problem

$$u_t - u = 5, \quad u(x,0) = 2x, \quad x > 0, \; t > 0.$$ [5 marks]

f. Find the general solution of the PDE

$$x^2 u_{xx} + xu_x - 4u = 0, \tag{1}$$

where $u = u(x,y)$. [5 marks]

Section B

Answer ANY 3 Questions in this section[1]

B.1 Consider the PDE

$$25u_{xx} + 20u_{xy} + 4u_{yy} = 24. \qquad (2)$$

i. Classify (2) as parabolic, hyperbolic or elliptic. [3 marks]

ii. Reduce (2) into its canonical form [12 marks]

iii. Find the general solution of (2). [5 marks]

B.2 Consider the Cauchy problem of the wave equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \quad -\infty < x < \infty,\ t \geqslant 0 \\ u(x,0) &= \Phi(x), \quad -\infty < x < \infty \\ u_t(x,0) &= \Psi(x), \quad -\infty < x < \infty. \end{aligned}$$

Derive the *d'Alembert solution*

$$u(x,t) = \frac{1}{2}\Big[\Phi(x+ct) + \Phi(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(\alpha)\mathrm{d}\alpha.$$ [20 marks]

B.3 Consider the Dirichlet problem for a circle

$$\begin{aligned} \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} &= 0, \quad 0 \le \rho \le a, \\ u(a,\phi) &= 1 - \cos\phi. \end{aligned}$$

Find the solution $u(\rho, \phi)$ of the problem. [20 marks]

B.4 a. Find the PDE satisfied by u, defined by

$$u = x^2 + y^2 + F(xy - u),$$

where $u = u(x,y)$ and F is an arbitrary function. [10 marks]

b. Find the solution of the PDE

$$u(xu_x - yu_y) = y^2 - x^2,$$

containing the straight line $u = x = y$. [10 marks]

---

[1]Please turn over for question B.5.

B.5 Solve the initial-boundary-value problem for a vibrating string. [20 marks]

$$\begin{aligned} c^2 u_{xx} - u_{tt} &= 0, & 0 < x < 2,\ t > 0 \\ u(0,t) = u(2,t) &= 0, & t \geqslant 0 \\ u(x,0) &= 1 - |x-1|, & 0 \leqslant x \leqslant 2 \\ u_t(x,0) &= 0, & 0 \leqslant x \leqslant 2. \end{aligned}$$

END OF EXAMINATION