University of ESwatini

Supplementary Examination, January 2019

B.A.S.S. , B.Sc, B.Ed

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Begin each major question (A1, A2, B2, etc) on a new page.
3. Show all your working.
4. Special requirements: None.
5. Indicate your program next to your student ID number.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: Answer All Questions

QUESTION A1

(a) Define;

i. An ideal N in a ring R.
ii. A divisor of zero in a ring R.
iii. The characteristic of a ring R.
iv. Unity in a ring R.
v. A unit in a ring R.

[10 Marks]

(b) Prove that in the ring $\mathbb{Z}_n$;

i. the divisors of zero are those elements that are not relatively prime to n.
ii. the elements that are relatively prime to n cannot be divisors of zero.

[10 Marks]

QUESTION A2

(a) Give a definition of the following:

i. an integral domain.
ii. a field.

[6 Marks]

(b) Prove that a finite domain is a field. [10 Marks]

(c) Give an example of an integral domain that is not a field. [4 Marks]

SECTION B: Answer Three(3) Questions Only

QUESTION B3

(a) Let $\phi_\alpha : \mathbb{Z}[x] \to \mathbb{Z}_7$ be the evaluation homomorphism. Evaluate each of the following;

i. $\phi_5\left[(x^3+2)(4x^2+3)(x^7+3x^2+1)\right]$.

ii. $\phi_4\left[3x^{106}+5x^{99}+2x^{53}\right]$.

[10 Marks]

(b) Show that the rings $\mathbb{Z}$ and $2\mathbb{Z}$ are not isomorphic. [4 Marks]

(c) Show that for a field F, the set of all matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ for $a, b \in F$ is a right ideal but not a left ideal of the ring $R = M_2(F)$. [6 Marks]

QUESTION B4

Which of the following are integral domains and which are fields? Justify your answer.

(a) $\mathbb{Z}_2 \times \mathbb{Z}_2$. [5 Marks]

(b) $\{a + b\, i : a, b \in \mathbb{Q}\}$. [5 Marks]

(c) $\mathbb{Z} \times \mathbb{R}$. [5 Marks]

(d) $\mathbb{R}[x]$. [5 Marks]

QUESTION B5

(a) Determine which of the following polynomials in $\mathbb{Z}[x]$ satisfy an Eisenstein's criterion for irreducibility over $\mathbb{Q}$.

i. $4x^{10} - 9x^3 + 24x - 18$.

ii. $2x^{10} - 25x^3 + 10x^2 - 30$.

[8 Marks]

(b) Express $f(x) = x^3 + 2x + 3$ in $\mathbb{Z}_5[x]$ as a product of irreducible polynomials in $\mathbb{Z}_5[x]$. [6 Marks]

(c) Prove that if D is an integral domain, then $D[x]$ is also an integral domain. [6 Marks]

QUESTION B6

(a) Let α be a zero of $x^2 + x + 1$ in the extension field of $\mathbb{Z}_2$. Give the addition and multiplication tables for the four elements of $\mathbb{Z}_2(\alpha)$. [6 Marks]

(b) Show that the polynomial $f(x) = x^p + a$ in $\mathbb{Z}_p[x]$ is not irreducible for any $a \in \mathbb{Z}_p$. [5 Marks]

(c) For each of the given algebraic numbers $\alpha \in \mathbb{C}$, find $\operatorname{irr}(\alpha, \mathbb{Q})$ and $\deg(\alpha, \mathbb{Q})$

i. $\sqrt{3 - \sqrt{6}}$.

ii. $\sqrt{\frac{1}{3} + \sqrt{7}}$

iii. $\sqrt{2} + i$

[9 Marks]

QUESTION B7

(a) Find all the monic irreducible polynomials of degree 2 over $\mathbb{Z}_3$. [9 Marks]

(b) Prove that every field is an integral domain. [7 Marks]

(c) Factor the polynomials $4x^2 - 4x + 8$ as a product of irreducibles in $\mathbb{Z}_{11}[x]$. [4 Marks]

END OF EXAMINATION