University of Eswatini

Main Examination, 2018/2019

B.A.S.S. II, B.Ed (Sec.) II, B.Sc. II, B.Eng. II

Title of Paper : Linear Algebra

Course Number : MAT221/M220

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

a) i) Express [6]

$$C = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

as a product of elementary matrices.

ii) Find A^7, where [2]

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) i) Find $|A|$ and $|A^{-1}|$, given that [4]

$$A = \begin{bmatrix} 2 & 4 & 4 & 6 \\ 0 & -1 & 0 & 9 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

ii) If $\det(A) = 0$, what can you conclude about the solutions of the linear system of equations $A\mathbf{x} = \mathbf{b}$? [2]

c) i) Determine the characteristic polynomial of [4]

$$A = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 3 & \alpha_2 & 0 \\ -4 & 0 & \alpha_3 \end{bmatrix}$$

ii) Hence find the corresponding eigenvalues. [4]

d) Define the following terms

i) linearly independent. [2]

ii) vector space. [4]

ii) Are the vectors $\bar{v}_1 = (2,5,3)$, $\bar{v}_2 = (1,1,1)$, and $\bar{v}_3 = (4,2,0)$ linearly independent? [4]

e) i) Solve the system using Gauss-Jordan elimination, where a, and b are constants. [6]

$$2x_1 + x_2 = a$$
$$3x_1 + 6x_2 = b$$

ii) Solve the system for $a = 1$, and $b = 0$. [2]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

a) What conditions must b_1, b_2 and b_3 satisfy in order for the system of equations to be consistent? [10]

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3. \end{aligned}$$

b) Prove the following theorems

i) If A is an invertible matrix, prove that A^T is also invertible and $(A^{-1})^T = (A^T)^{-1}$. [6]

ii) Every elementary matrix is invertible and the inverse is also an elementary matrix. [4]

QUESTION B3 [*20 Marks*]

a) If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$. [5]

b) Consider the linear system of equations

$$\begin{aligned} kx_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_2 + (1+k)x_3 &= 2. \end{aligned}$$

i) By analyzing the determinant of the coefficient matrix, determine the value of k for which the system have exactly one solution. [5]

ii) Find the solution(s) of the system for $k = 1$. [10]

QUESTION B4 [*20 Marks*]

a) Verify Cayley-Hamilton theorem for [6]

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

b) Find bases for the eigenspaces of the matrix [6]

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

c) Prove that a square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A. [8]

QUESTION B5 [*20 Marks*]

a) Let M_{nn} be the vector space of $n \times n$ matrices. In each part determine whether the transformation is linear.

   i) $T_1(A) = A^T$ [4]

   ii) $T_2(A) = \det(A)$. [4]

b) Prove that If $T : V \to W$ is a linear transformation, then:

   i) $T(\mathbf{0}) = \mathbf{0}$ [2]

   ii) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$. [4]

c) Define a linear transformation T from V to W. [6]

QUESTION B6 [*20 Marks*]

a) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. Prove that if $r > n$, then S is linearly dependent. [6]

b) Determine whether the polynomials

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly dependent or linearly independent in $\mathbb{P}_2$. [7]

c) Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, $\mathbf{v}_3 = (3, 3, 4)$, form a basis for $\mathbb{R}^3$. [7]

END OF EXAMINATION PAPER