UNIVERSITY OF ESWATINI

DECEMBER 2018 MAIN EXAMINATION

BSc III, B.Ed III, BASS III, BEng IV

Title of Paper : Numerical Analysis I

Course Number : MAT311/M311

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Let $f(x) = \sqrt{1+x} - \sqrt{1-x}$. Rewrite this function to an equivalent form that seeks to avoid loss of significance when x is small. [3 Marks]

(b) Consider the linear system

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 2 \\ 5 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

Starting from the initial approximation $x_0 = (0, 0, 0)$, use two (2) iterations of the Gauss-Siedel iteration method to approximate the solution. [7 Marks]

(c) Use the Newton divided-difference formula to compute the cubic interpolating polynomial $P_3(x)$ for the function $f(x) = 2^x$ using interpolation points

$$x_0 = 0,\ x_1 = 1,\ x_2 = 2 \text{ and } x_3 = 3$$

. [7 Marks]

(d) Use the quadrature formula

$$\int_{-1}^{1} f(x)\, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

to approximate the integral $\int_{-1}^{1} x^2 e^x\, dx$ [3 Marks]

(e) Use the Naive Gaussian Elimination Method to find the LU factorisation of the matrix A in which L is a unit lower triangular matrix and U is an upper triangular matrix

$$\begin{bmatrix} 4 & -6 & -5 \\ 4 & -5 & -5 \\ -8 & 2 & -6 \end{bmatrix}$$

[8 Marks]

(f) Derive the approximation formula

$$f'(x) \approx \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)]$$

and show that its error term is $\frac{h^2}{3} f'''(\xi)$ [6 Marks]

(g) Starting with $x_0 = 0$, perform two iterations to find an approximate root of the equation $x^3 - 4x + 1 = 0$, using the Newton's method [6 Marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 Consider the function

$$f(x) = e^{-x^2} - x = 0$$

that is to be solved in the interval $[0, 1]$.

(a) Show that the function has exactly one root in the interval $[0, 1]$. [4 marks]

(b) State the fixed point theorem [4 Marks]

(c) Show that $x = g(x) = e^{-x^2}$ has a unique fixed point in [0,1] for any initial guess $x_0 \in [0, 1]$. [4 Marks]

(d) Show that the Newton's method iteration scheme for finding the root of $f(x)$ is

$$x_{n+1} = \frac{2x_n^2 + 1}{2x_n + e^{x_n^2}}$$

[4 Marks]

(e) Starting from $x_0 = 0$, apply four iterations of the Newton's method to approximate the root of $f(x) = 0$. [4 Marks]

QUESTION B3 [*20 Marks*]

B3 (a) Given that the standard single-precision floating point representation of a machine number is

$$(-1)^s \times (1.f)_2 \times 2^{c-127}$$

Determine the machine representation of the number -125.234375. [7 Marks]

(b) Determine the decimal number that has the following single precision representation

1	10001011	00000011011100000000000

[7 Marks]

(c) How can accurate values of the function

$$f(x) = x - \sin x$$

be computed near $x = 0$? [6 Marks]

QUESTION B4 [*20 Marks*]

B4 (a) Use Taylor series expansions to derive the following difference formula

$$f'(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

[6 Marks]

(b) Let $f(x) = \sin x$, where x is measured in *radians*. Calculate approximations to $f'(0.8)$ using the formula in (a) with $h = 0.1$ (Work with six decimal places) [4 Marks]

(c) The 3-Point formulas for approximating $f'(x_0)$ are given by

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)]$$
$$f'(x_0) = \frac{1}{2h}[-f(x_0-h) + f(x_0+h)]$$
$$f'(x_0) = \frac{1}{2h}[f(x_0-2h) - 4f(x_0-h) + 3f(x_0)]$$

Given the following times and distances

Time	0	3	5	8	10	13
Distance	0	225	383	623	742	993

Use $h = 5$ and the three-point formulas above to show that the speeds at each time are given by

Time	0	3	5	8	10	13
Speed	79	82.4	74.2	76.8	69.4	71.2

[10 Marks]

QUESTION B5 [*20 Marks*]

B5 (a) Find the LU factorisation of the matrix A in which L is a unit lower triangular matrix and U is an upper triangular matrix

$$A = \begin{bmatrix} 4 & -6 & -5 \\ 4 & -5 & -5 \\ -8 & 2 & -6 \end{bmatrix}$$

[12 Marks]

(b) Use the LU factorization in (a) to solve the linear system

$$4x_1 - 6x_2 - 5x_3 = -50$$
$$4x_1 - 5x_2 - 5x_3 = -47$$
$$-8x_1 + 2x_2 - 6x_3 = 6$$

[8 Marks]

QUESTION B6 [*20 Marks*]

B6 (a) Use the Trapezoid method to estimate the integral $\int_0^1 e^{1-x^2}$ using a step size of $h = 0.25$. [4 Marks]

(b) Determine a formula of the form

$$\int_0^h f(x)\,dx = w_0 f(0) + w_1 f(h) + w_2 f''(0) + w_3 f''(h)$$

that is exact for polynomials of as high a degree as possible. [6 Marks]

(c) Construct a quadrature rule on the interval $[0, 4]$ using the nodes $1, 2, 3$. [8 Marks]

(d) Use the quadrature formula derived in (c) above to compute the integral

$$\int_0^4 \frac{1}{1+x^2}$$

[2 Marks]

END OF EXAMINATION PAPER