University of Eswatini

Main Examination, 2018/2019

BSc III, BEng III, BEd III, BASS III

Title of Paper : VECTOR ANALYSIS

Course Number : MAT312/M312

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Answer ANY THREE (3) questions in this Section B. Each question in this section is worth 20%.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A

Answer ALL Questions in this section

A.1 a. Evaluate

i. $\int_0^{2\pi} \sin^4\theta \cos^6\theta \mathrm{d}\theta$ [5 marks]

ii. $\int_{-\infty}^{\infty} x^2 e^{-x^2} \mathrm{d}x$ [5 marks]

b. Find the work done in moving an object

i. along the straight line from $(2,-4,5)$ to $(-5,0,8)$ in the force field $\boldsymbol{F} = \langle 3,-2,-9\rangle$. [3 marks]

ii. from $(1,2)$ to $(3,18)$ along the parabola $y = 2x^2$ in the force field $\boldsymbol{F} = (3x^2+y)\hat{\boldsymbol{\imath}} + (x-1)\hat{\boldsymbol{\jmath}}$. [5 marks]

c. Find the *parametric equation* of the straight line passing through $(4,3,-2)$ and $(-6,4,5)$. [3 marks]

d. Find the *scalar equation* of the plane passing through the point $(-6,5,-4)$ and parallel to the vectors $-3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}$ and $7\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}}$. [5 marks]

e. Given the surface defined by

$$z = 4 + xe^{4x^2-y^2}.$$

i. Find the *upward-pointing* normal vector of the surface at $(-1,2,3)$. [5 marks]

ii. Hence, or otherwise, find the equation of the tangent plane to the surface at $(-1,2,3)$. [3 marks]

f. i. State the *Divergence Theorem.* [2 marks]

ii. Hence, or otherwise, evaluate $\iint_S \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S}$ where $\boldsymbol{F} = \langle 6x^2, -2y, z\rangle$ and S is the closed surface bounding the cube with vertices $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$, $(0,0,2)$, $(2,0,2)$, $(2,2,2)$, $(0,2,2)$. [4 marks]

Section B

Answer ANY Three (3) Questions in this section

B.2 a. The Laguerre polynomials are defined by the Rodrigue's formula

$$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n e^{-x}\right).$$

Use this formula to find

i. $L_1(x)$ [2 marks]

ii. $L_2(x)$ [3 marks]

b. Alternatively, the Laguerre polynomials are defined by the generating function

$$G(x,t) = \frac{\exp\left(\frac{-xt}{1-t}\right)}{1-t} = \sum_{n=0}^{\infty} t^n L_n(x).$$

By differentiating the generating function with respect to x, derive the recurrence relation

$$L'_{n+1}(x) = L'_n(x) - L_n(x).$$ [8 marks]

c. Prove that

$$2^{2n}B(n, n+1) = B(n, \tfrac{1}{2})$$

where B is the Beta function. [7 marks]

B.3 Consider the two straight lines

$$\ell_1 \;:\; x = 2+4t,\ y = 2-t,\ z = 8+3t,\ \ t \in \mathbb{R}$$
$$\ell_2 \;:\; \frac{x+2}{-2} = \frac{y-3}{4} = 5-z.$$

a. Find the shortest distance from ℓ_1 to the point $(1,7,4)$, leaving your answer in surd form. [6 marks]

b. Find the point of intersection of ℓ_1 and ℓ_2. [6 marks]

c. Find the angle of intersection between ℓ_1 and ℓ_2 (in degrees, correct to 1 decimal point) [3 marks]

d. Find the equation of the plane containing both ℓ_1 and ℓ_2. [5 marks]

B.4 Verify Stokes' theorem where C is the curve of intersection between the plane $4x + z = 12$ and the cylinder $x^2 + y^2 = 4$, and the vector function $\boldsymbol{F} = \langle -y, x, z \rangle$. [20 marks]

B.5 a. Use vector methods to prove the cosine formula

$$c^2 = a^2 + b^2 - 2ab\cos C$$

for a scalene triangle with sides of length a, b and c, respectively, where C is the acute angle between sides a and b. [6 marks]

b. Given the vector function

$$\boldsymbol{F} = F_1(x, y, z)\hat{\boldsymbol{\imath}} + F_2(x, y, z)\hat{\boldsymbol{\jmath}} + F_3(x, y, z)\hat{\boldsymbol{k}}$$

where F_1, F_2 and F_3 are twice differentiable functions, prove that

$$\nabla \cdot \left(\nabla \times \boldsymbol{F}\right) = 0.$$ [7 marks]

c. Consider two parallel lines ℓ_1 and ℓ_2 with direction vector $\boldsymbol{m}$, one passing through $P_1(x_1, y_1, z_1)$ and the other passing through $P_2(x_2, y_2, z_2)$, with equations

$$\begin{aligned} \ell_1 : \boldsymbol{r} &= \boldsymbol{r}_1 + \boldsymbol{m}t, \quad t \in \mathbb{R} \\ \ell_2 : \boldsymbol{r} &= \boldsymbol{r}_2 + \boldsymbol{m}s, \quad s \in \mathbb{R}, \end{aligned}$$

where t and s are parameters, and $\boldsymbol{r}_1 = \langle x_1, y_1, z_1 \rangle$ and $\boldsymbol{r}_2 = \langle x_2, y_2, z_2 \rangle$ are the position vectors of P_1 and P_2, respectively. Prove that the shortest distance between ℓ_1 and ℓ_2 is given by

$$\rho = \left| \frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{m}}{|\boldsymbol{m}|} \right|.$$ [7 marks]

B.6 a. Consider the vector function

$$\boldsymbol{F} = 2xy\hat{\boldsymbol{\imath}} + \left(x^2 - z^2 \sin y\right)\hat{\boldsymbol{\jmath}} + 2z\cos y\hat{\boldsymbol{k}}.$$

i. Prove that $\boldsymbol{F}$ is a conservative force field. [4 marks]

ii. Find a potential function Φ such that $\boldsymbol{F} = \nabla\Phi$ [8 marks]

b. Find the flux of $\boldsymbol{F} = \langle 2x, 3y, 3z \rangle$ through S the closed surface made up of the cone $z = 3 - \sqrt{x^2 + y^2}$ above, and the disk $x^2 + y^2 \leqslant 9$ below. [8 marks]

END OF EXAMINATION