University of Eswatini

Resit/Supplementary Examination – July 2019

BSc III, BEd III, BASS III, BEng III

Title of Paper	:	Vector Analysis
Course Number	:	MAT312/M312
Time Allowed	:	Three (3) hours

Instructions:

1. This paper consists of 2 sections.
2. Answer ALL questions in Section A.
3. Answer ANY 3 questions in Section B.
4. Show all your working.
5. Begin each question on a new page.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A

Answer ALL Questions in this section

A.1 a. Evaluate

$$\int_0^\infty \frac{x^2\,\mathrm{d}x}{1+x^6}.$$ [5 marks]

b. Determine whether the set of functions $\{x,\ \ x^2-2x+3\}$ is orthogonal over the interval $[0,\infty)$ with respect to the weight function $w(x)=e^{-x}$. [5 marks]

c. Evaluate the line integral

$$\int_C (8y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}) \cdot (\mathrm{d}x\hat{\boldsymbol{\imath}} + 8\mathrm{d}y\hat{\boldsymbol{\jmath}})$$

where C is

i. the straight line from $(1,1)$ to $(-1,1)$ [3 marks]

ii. the arc along the circle $x^2+y^2=2$ from $(1,1)$ to $(-1,1)$ in the counterclockwise direction [5 marks]

d. Determine whether the following set of vectors is coplanar.

$$\boldsymbol{u}=\langle 2,5,-2\rangle,\ \ \boldsymbol{v}=\langle 3,-1,0\rangle,\ \ \boldsymbol{w}=\langle 5,9,-4\rangle.$$ [5 marks]

e. Make a sketch of the 2-dimensional vector flow field of

$$\boldsymbol{F}=y\hat{\boldsymbol{\imath}}-\hat{\boldsymbol{\jmath}}.$$ [5 marks]

f. List 3 properties of conservative vector fields. [5 marks]

g. i. State *Green's Theorem in the Plane.* [2 marks]

ii. Use the area formula

$$A=\frac{1}{2}\oint_C x\mathrm{d}y - y\mathrm{d}x$$

to find the area of the ellipse centred at the origin, with major axis 16 and minor axis 10. [5 marks]

Section B

Answer ANY Three (3) Questions in this section

B.2 a. Prove that

$$\int_{-1}^{1}(1-x)^{m-1}(1+x)^{n-1}\mathrm{d}x = 2^{m+n-1}B(m,n).$$ [7 marks]

Hence, or otherwise, evaluate

$$\int_{-1}^{1}(1-x)^{3}\sqrt{1+x}\,\mathrm{d}x.$$ [3 marks]

b. The Hermite polynomials are defined by the Rodrugue's formula

$$H_n(x) = e^{x^2}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n e^{-x^2}.$$

Use the Rodrigue's formula to derive the recurrence relations

i. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ [6 marks]

ii. $H_n'(x) = 2nH_{n-1}(x)$ [4 marks]

B.3 a. Consider the straight line ℓ and plane Π :

$$\ell \ : \ \frac{x-5}{2} = \frac{y+3}{6} = \frac{z-4}{-3}$$
$$\Pi \ : \ x - 2y + 3z = -15.$$

i. Find the point of intersection of ℓ and Π. [4 marks]

ii. Find the angle of intersection between ℓ and Π. [3 marks]

iii. Find the shortest distance from the origin to ℓ. [5 marks]

b. Consider the planes

$$\Pi_1 \ : \ 2x - y + z = 3$$
$$\Pi_2 \ : \ x + 2y - z = -21.$$

i. Find the line of intersection of Π_1 and Π_2, expressing it in parametric form. [5 marks]

ii. Find the angle of intersection between Π_1 and Π_2. [3 marks]

B.4 Given that $\boldsymbol{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}}$ and $r = \sqrt{x^2 + y^2 + z^2}$, show that

a. $\nabla(\ln r) = \dfrac{\boldsymbol{r}}{r^2}$ [5 marks]

b. $\nabla^2(\ln r) = \dfrac{1}{r^2}$ [5 marks]

c. $\nabla \cdot (r^4 \boldsymbol{r}) = 7r^4$ [5 marks]

d. $\nabla \times (r^2 \boldsymbol{r}) = 0$ [5 marks]

B.5 a. Find the upward flux of $\boldsymbol{F} = (x + y^2)\hat{\boldsymbol{\imath}} - 2x\hat{\boldsymbol{\jmath}} + 2yz\hat{\boldsymbol{k}}$ through the plane $2x + y + 2z = 12$ in the first octant. [10 marks]

b. Find the surface area of the paraboloid of revolution $z + 4(x^2 + y^2) = 16$ above the xy-plane. [10 marks]

B.6 Verify the divergence theorem for the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the plane $z = 5$, where $\boldsymbol{F} = 4x\hat{\boldsymbol{\imath}} + 4y\hat{\boldsymbol{\jmath}} + 8z\hat{\boldsymbol{k}}$. [20 marks]

END OF EXAMINATION