UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2018/2019

BASS III, B.Ed (Sec.) III, B.Sc. III, B.Eng. III

Title of Paper : Complex Analysis

Course Number : MAT313/M313

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

a) Evaluate the following and leave your answer in the form $a+ib$.

i) $\ln(ie)$. [4]

ii) $\cosh\left(2\pi i - \frac{1}{2}\right)$. [4]

b) i) Determine whether $f(z) = \frac{z^3}{z^3+3z^2+z}$ is continuous at the point $z_0 = i$. [3]

ii) Determine whether $f(z) = r^2 e^{-2i\theta+\pi}$ is holomorphic or not. [5]

c) i) Evaluate $\int_C \frac{e^z}{z-2}dz$ if C is given by $|z| = 3$. [2]

ii) Let $f(z)$ be continuous on a domain D. If $\int_C f(z)dz = 0$, for every closed contour C lying in D, what can we conclude about $f(z)$ throughout D? [2]

d) i) Find the Maclaurin series of $\Omega(z) = z^3 e^{4z^2}$. [4]

ii) What is the main difference between a Laurent series and a Taylor series? [4]

e) i) Find and classify the singularities of $f(z) = \frac{1}{z^2+4}$ in the upper half plane. Find the corresponding residue. [4]

ii) Using your answer in part *i*), evaluate $\int_C \frac{dz}{z^2+4}$ where C is a semi-circle in the upper half plane of radius six. [4]

iii) Find $\int_C (4\bar{z} - 3z)dz$, where C is the right-hand half of the circle $|z| = 2$, from $z = -2i$ to $z = 2i$. [4]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

a) Evaluate $Ln(1+i)$ express your answer in the form $a+ib$. [2]

b) Consider the equation $\cos(z) = 2i\sin(z)$. Solve for z. [8]

c) Show that $\sinh^{-1}(z) = \ln(z + \sqrt{z^2+1})$ [10]

QUESTION B3 [*20 Marks*]

a) Determine if $g(z) = 4z - 6\bar{z} + 3$ is a regular function or not. [4]

b) Verify that the function, $\alpha(x,y) = \sinh(x)\sin(y)$ is harmonic and find their harmonic conjugate $\beta(x,y)$ and the analytic function $\Omega(z) = \alpha(x,y) + i\beta(x,y)$. [6]

c) Prove that if a function $\phi(z) = \alpha(x,y) + i\beta(x,y)$ is analytic in a domain D, then $\alpha(x,y)$ and $\beta(x,y)$ are harmonic in D. [10]

QUESTION B4 [*20 Marks*]

a) Evaluate $\int_C \frac{e^z - 4z^2}{(z-2)^4} dz$ if C is [10]

i) the circle $|z+3| = 9$

ii) the circle $|z - 8 - 4i| = 4$

b) Let $f(z)$ be analytic in the simply connected domain D containing the circular contour C of radius ρ centered at z_0. If at each point z on C, $|f(z)| \leq \Gamma$, prove that [10]

$$|f^{(n)}(z_0)| \leq \frac{n!\Gamma}{\rho^n}, \quad \text{for}, \quad n = 1, 2, \cdots.$$

QUESTION B5 [*20 Marks*]

a) Determine if the sequence $z_n = \frac{7}{n^8} - 6i$ for $n = 1, 2, \cdots$ converges or diverges. [4]

b) Show that $\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$. [8]

c) Find the Laurent series that represents the function $f(z) = \frac{z}{(2-z)(z-1)}$ in the domain $1 < |z| < 2$. [8]

QUESTION B6 [*20 Marks*]

a) Consider the function $f(z) = \frac{z^2+4}{z^2-2z+5}$.

i) Locate and classify all singularities. [4]

ii) Find the value of the residue at each singularity. [2]

iii) Hence evaluate $\int_C \frac{z^2+4}{z^2-2z+5}$, where C is the contour defined by $|z-1| = 4$ [6]

b) Using Cauchy's Residue Theorem, evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}.$$

[8]

END OF EXAMINATION PAPER