University of Eswatini

Main Examination, 2018/2019

BASS III, B.Ed. (Sec.) III; B.Sc III

Title of Paper : Abstract Algebra I

Course Number : M323/MAT324

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

This examination paper should not be opened until permission has been given by the invigilator.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Define each of the following:

i. A relation from a set X into a set Y. [2 marks]

ii. A mapping from a set X into a set Y. [2 marks]

iii. A binary operation on a set X. [3 marks]

iv. If $\mathbb{Z}$ denote set of integers, let $*$ be a binary operation on $\mathbb{Z}$ defined by $x * y = xy + 4$ for all $x, y \in \mathbb{Z}$.
(i) Determine $(4 * -2) * 5$, (ii) If $x * 2 = 10$, find x. [4 marks]

(b) i. State Principle of Well-Ordering. [2 marks]

ii. Is it possible to pay total of $E100674$ for buying several $E12$ items and several $E32$ items? [5 marks]

(c) i. Give the definition of a group. [5 marks]

ii. Let $(\mathbb{Z}, \oplus)$ be a group, where $x \oplus y = x + y - 1$ for all $x, y \in \mathbb{Z}$. Find the identity element of $\mathbb{Z}$ and inverse of each of the element under the operation $\oplus$. [4 marks]

iii. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. Compute $\alpha^{-1}\beta$. [3 marks]

iv. Write the following permutations in the cyclic notation.
$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$ [4 marks]

(d) i. Define a subgroup of a group. [2 marks]

ii. State Lagrange's Theorem. [2 marks]

iii. State Fundamental Homomorphism Theorem. [2 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Let $\mathbb{Z}$ be a set of integers. For any $a, b, c \in \mathbb{Z}$, prove that $c \mid (xa + yb)$ if $c \mid a$ and $c \mid b$ for all $x, y \in \mathbb{Z}$. [9 marks]

(b) Let $\mathbb{N}$ be set of natural numbers. Prove that for all $n \in \mathbb{N}$, $27 \mid (10^n + 18n - 1)$. [11 marks]

QUESTION B3 [*20 Marks*]

B3 (a) Let $\mathbb{Q}$ be a set of rational numbers. Define a binary operation $\star$ on $G := \mathbb{Q} - \{0\}$ by

$$a \star b = \frac{ab}{3} \quad \text{for all} \quad a, b \in G.$$

Show that $(G, \star)$ is a group. [11 marks]

(b) Prove that a group $(G, *)$ is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. [9 marks]

QUESTION B4 [*20 Marks*]

B4 (a) Define the order of an element of a group G. [4 marks]

(b) If $G = (\{1, -1, i, -i\}, \cdot)$ where $i = \sqrt{-1}$. Find the order of -1 and i. [6 marks]

(c) Find the order of $\gamma = (1\ 2\ 5\ 8\ 13)(349)(10\ 12) \in S_{13}$ and hence express γ^{245} in cycle notation. [10 marks]

QUESTION B5 [*20 Marks*]

B5 (a) Define a normal subgroup H of a group G. [5 marks]

(b) If $G = S_3,\ H = \langle(12)\rangle = \{e, (12)\}$. Prove that H is not a normal subgroup of G. [6 marks]

(c) Let H be a normal subgroup of a group G and K be any subgroup of G. Prove that $HK = \{hk : h \in H,\ k \in K\}$ is a subgroup of G. [9 marks]

QUESTION B6 [*20 Marks*]

B6 (a) Let $(G, *)$ and $(H, \odot)$ be two groups. Define a homomorphism from $(G, *)$ to $(H, \odot)$. [4 marks]

(b) Show that a mapping $\beta : (\mathbb{R}, +) \to (\mathbb{R} - \{0\}, \cdot)$ defined by $\beta(x) = 3^x$ for all $x \in \mathbb{R}$ is a homomorphism. [4 marks]

(c) Let $\alpha : G \to G'$ be a group homomorphism. Prove that kernel of α, denoted by $Ker(\alpha)$ is a normal subgroup of G. [12 marks]

END OF EXAMINATION