University of Eswatini

Final Examination, December 2018

B.Sc III, B.A.S.S III, B.Ed III

Title of Paper : Real Analysis

Course Code : MAT331/M331

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.

   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.

   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.

2. Each question in Section B is worth 20%.

3. Show all your working.

4. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

Question 1

(a) Define the following terms

(i) Supremum and infimum of a set A. [2]

(ii) Subsequence. [2]

(iii) Uniformly continuous functions. [2]

(iv) Cauchy sequence. [2]

(b) (i) Prove that every finite set is bounded. [4]

(ii) Let S be a set. Prove that supremum of S if it exists, is unique. [4]

(iii) Give the ϵ, N definition for the convergence of a sequence $< a_n >$ to a number L. [4]

(iv) Show that every convergent sequence has a unique limit. [4]

(v) If $a_n = 2 + \frac{(-1)^n}{n^2}$, find the least positive integer m such that $|a_n - 2| < \frac{1}{10^4} \quad \forall n > m$. [4]

(vi) If a function f is uniformly continuous on an interval I, then show that it is continuous on I. [4]

(vii) Prove that if a function is differentiable at a point then it is continuous at that point. [4]

(viii) Let $f(x) = x$ for $x \in [0, 1]$ and let $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ be a partition of $[0, 1]$. Compute $\underline{\mathscr{D}}(P, f)$ and $\overline{\mathscr{D}}(P, f)$. [4]

SECTION B: ANSWER ANY 3 QUESTIONS

Question 2

(a) Discuss the boundedness of the sequence $a_n = \dfrac{n}{n+1}$. [6]

(b) Prove that the sequence whose n^{th} term is $a_n = \sqrt{n+1} - \sqrt{n}$ is monotonic. [6]

(b) Prove that every monotonically increasing sequence which is bounded above converges to its least upper bound. [8]

Question 3

(a) (i) TRUE or FALSE. If a sequence $< a_n >$ converges to l, then every subsequence of $< a_n >$ also converges to l. [2]

(ii) If a sequence has a convergent subsequences, then the sequence is convergent. TRUE or FALSE? Explain your answer. [4]

(b) Examine the sequence $a_n = \frac{(-1)^n}{n}$ for cluster points. [6]

(c) Prove, by definition, that the sequence whose terms are given by $\dfrac{1}{n^2}$ is a Cauchy sequence. [8]

Question 4

(a) Test for the convergence of the series $\dfrac{1^2}{2^2} + \dfrac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \dfrac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$ [10]

(b) Show that the necessary and sufficient condition for the convergence of a positive term series $\sum u_n$ is that a sequence $< S_n >$ of its partial sums is bounded. [10]

Question 5

(a) Using $\epsilon - \delta$ definition, show that

$$f(x) = \begin{cases} \frac{x^3-1}{x^2-1} & if x \neq 1 \\ \frac{3}{2} & if x = 1 \end{cases}$$ is continuous at $x = 1$ [10]

(b) Show that the function defined by $f(x) = x^3$ is uniformly continuous on $[-2, 2]$.

[10]

Question 6

(a) Let $f(x) = x^2$ for $x \in [0, 1]$ and let $P = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ be a partition of $[0, 1]$. Compute $\overline{\mathscr{D}}(P, f)$ and $\underline{\mathscr{D}}(P, f)$. [10]

(b) Let $f : [-1, 1] \to \Re$ be given by

$$f(x) = |x| = \begin{cases} -x & \text{when } x \leq 0 \\ x & \text{when } x > 0 \end{cases}$$

Show that f is Riemann integrable and find $\int_{-1}^{1} f(x)dx$. [10]

End of Examination Paper