# UNIVERSITY OF SWAZILAND

MAIN EXAMINATION, 2018/2019

## BSc.III,B.Ed III, BASS III

**Title of Paper** : Mathematical Statistics I

**Course Number** : MAT340

**Time Allowed** : Three (3) Hours

**Instructions**

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

**Special Requirements: NONE**

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

## SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

### QUESTION A1 [*40 Marks*]

A1 (a) Two football teams M and C each have one game left to play (not against each other) in the season. If M wins and C does not win, or if M draws and C loses, then M wins the championship. Otherwise C wins the championship. The probabilities that M wins, draws or loses the last game are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$, respectively. The probabilities that C wins, draws or loses the last game are $\frac{2}{3}$, $\frac{1}{6}$ and $\frac{1}{6}$, respectively.

(i) What is the probability that M wins the championship?

[5 Marks]

(ii) What is the probability that C has drawn the last game given that M has won the championship?

[3 Marks]

(b) Suppose that

$$F(y) = \begin{cases} 0, & \text{for } y < 0 \\ y, & \text{for } 0 \le y \le 1 \\ 1, & \text{for } y > 1 \end{cases}$$

Find the probability density function for $Y$ and compute $Var(Y)$.

[6 Marks]

(c) Consider an experiment where two dice are rolled. Let $A$ be the event where the sum of two dice equals 3, $B$ be the event that the sum of two dice equals 7 and $C$ be the event that at least one of the dice shows a 1. Compute $P(A|C)$. Are $A$ and $C$ independent?

[8 Marks]

(d) Let $X$ have a geometric distribution with parameter $p$. Compute $E(X)$.

[6 Marks]

(e) Suppose that a random system of security patrol is devised so that a patrol officer may visit a given location $Y = 0, 1, 2, 3, \ldots$ times per half-hour period, with each location being visited an average of once per time period. Assume that $Y$ possesses, approximately, a Poisson probability distribution. Calculate the probability that the patrol officer will miss a given location during a half-hour period. What is the probability that it will be visited at least once?

[5 Marks]

(f) Suppose that $Y$ has an exponential probability density function. Show that, if $a > 0$ and $b > 0$,

$$P(Y > a + b|Y > a) = P(Y > b).$$

[7 Marks]

## SECTION B: ANSWER ANY *THREE* QUESTIONS

### QUESTION B2 [*20 Marks*]

B2 (a) The joint probability distribution function of two random variables $Y$ and $Z$ is

$$P(Y = y, Z = z) = \frac{e^{-\lambda}\lambda^{y+z}}{y!z!}\theta^y(1-\theta)^z, \quad y = 0, 1, 2, \ldots; z = 0, 1, 2, \ldots$$

(i) Find the marginal distribution of $Y$ and identify it.

[4 Marks]

(ii) Show that $Y$ and $Z$ are independent.

[8 Marks]

(b) Find the moment-generating function for a gamma-distributed random variable.

[8 Marks]

### QUESTION B3 [*20 Marks*]

B3 (a) Let $Y_1, Y_2, \ldots, Y_n$ denote independent random variables with cumulative distribution function $F(y)$ and probability density function $f(y)$.

(i) Derive the probability density function of $Y_{(n)} = max\{Y_1, Y_2, \ldots, Y_n\}$.

[6 Marks]

(ii) Electronic components of a certain type have a length of life $Y$, with probability density given by

$$f(y) = \begin{cases} (1/100)e^{-y/100}, & \text{if } y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

(Length of life is measured in hours.) Suppose that two such components operate independently and in parallel in a certain system (hence, the system does not fail until both components fail). Find the density function for $X$, the length of life of the system. Hence compute the probability that $X > 200$ hours.

[6 Marks]

(b) Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with $E(Y_i) = \mu$ and $V(Yi) = \sigma^2$. Let

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

and show that $E(\bar{Y}) = \mu$ and $Var(\bar{Y}) = \sigma^2/n$.

[8 Marks]

**QUESTION B4** [*20 Marks*]

B4 (a) A soft-drink machine has a random amount $Y_2$ in supply at the beginning of a given day and dispenses a random amount $Y_1$ during the day (with measurements in gallons). It is not resupplied during the day, and hence $Y_1 \leq Y_2$. It has been observed that $Y_1$ and $Y_2$ have a joint density given by

$$f(y_1, y_2) = \begin{cases} 1/2, & \text{if } 0 \leq y_1 \leq y_2 \leq 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the conditional probability density of $Y_1$ given $Y_2 = y_2$. Evaluate the probability that less than 1/2 gallon will be sold, given that the machine contains 2 gallons at the start of the day.

[10 Marks]

(a) Consider the experiment of tossing a fair coin 3 times. Let $X$ be the number of heads on the first toss and $F$ the number of heads on the first two tosses. Fill the joint probability table for $X$ and $F$. Compute Cov(X, F).

[10 Marks]

**QUESTION B5** [*20 Marks*]

B5 (a) The median of the distribution of a continuous random variable Y is the value $\phi_{0.5}$ such that $P(Y \leq \phi_{0.5}) = 0.5$. What is the median of the uniform distribution on the interval $(\theta_1, \theta_2)$?

[5 Marks]

(b) The discrete random variable $X$ has the binomial distribution

$$P(X = x) = \binom{m}{x} \theta^x (1-\theta)^{m-x}, \quad x = 0, 1, \ldots, m$$

where $m$ is a positive integer and $0 < \theta < 1$. Find the moment-generating function for $X$ and use it to find the expected value and variance.

[10 Marks]

(c) $X_1, X_2, \ldots, X_n$ are independent random variables, each with the binomial distribution given above. Use moment generating functions to prove that $S = X_1 + X_2 + \ldots + X_n$ is also a binomial random variable.

[5 Marks]

**QUESTION B6 [*20 Marks*]**

B6 (a) The continuous random variables $X$ and $Y$ have joint probability density function

$$f_{XY}(x,y) = \begin{cases} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} x^{\alpha-1}y^{\beta-1}(1-x-y)^{\gamma-1}, & 0 < x < 1, 0 < y < 1, x+y < 1, \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ are parameters and $\Gamma(.)$ is the gamma function. Obtain the joint probability density function of

$$U = 1 - X, \ V = \frac{Y}{1-X}.$$

[10 Marks]

(b) Obtain the marginal probability density functions for $U$ and $V$ and identify these marginal distributions. Compute $E(V)$.

[10 Marks]

END OF EXAMINATION PAPER