UNIVERSITY OF ESWATINI

DECEMBER 2018 MAIN EXAMINATION

BSc IV, B.Ed IV, BASS IV

Title of Paper : Numerical Analysis II

Course Number : MAT411/M411

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Determine if the initial value problem

$$y' = 1 + t\sin(ty), \quad 0 \le t \le 2, \ y(0) = 0$$

has a unique solution for $0 \le t \le 2$ [5 Marks]

(b) What constant a would make the equation

$$\sum_{k=0}^{m} [f(x_k) - ae^{x_k}]^2$$

as small as possible [5 Marks]

(c) Find the linear least squares approximation of

$$f(x) = x^3 + x - 1$$

on the interval $[0, 1]$. [6 Marks]

(d) Apply the improved Euler's method with $h = 0.1$ to solve the initial value problem

$$y' = t + \frac{3y}{t}, \quad y(1) = 0$$

up to $y(1.2)$ and compare the approximate solution against the exact solution and compare them with the exact solution $y(t) = t^3 - t^2$ [5 Marks].

(e) Consider the following multi-step method

$$y_{n+1} = y_{n-1} + \frac{h}{3}\{f_{n-1} + 4f_n + f_{n+1}\}$$

i. Is the method implicit or explicit? [1 Mark]

ii. Prove that the method is of order 4 and find the leading term in the local truncation error [6 Marks

(f) Use the 3-step Adams-Bashforth method

$$y_{i+1} = y_i + \frac{h}{12}[23f(t_i, y_i) - 16f(t_{i-1}, y_{i-1}) + 5f(t_{i-2}, y_{i-2})]$$

with $h = 0.2$ to approximate $y(0.8)$ if

$$\frac{dy}{dt} = t - y, \quad y(0) = 1, \ y(0.2) = 0.837462, \ y(0.4) = 0.7406401$$

[5 Marks]

(g) Use finite differences with step size $h = \frac{1}{3}$ and central difference approximation on all derivatives to approximate the solution of the boundary value problem

$$y'' - xy' + 3y = 10x, \quad y(0) = 0, \ y(1) = 3$$

[7 Marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Use the method of undetermined coefficients to derive the 2-step Adams-Moulton formula

$$y_{i+1} = y_i + \frac{h}{12}[5f(t_{i+1}, y_{i+1}) + 8f(t_i, y_i) - f(t_{i-1}, y_{i-1})]$$

[7 Marks]

(b) Determine the local truncation error of 2-step Adams-Moulton method. [6 Marks]

(c) Analyse the stability, consistency and convergence of the 2-step Adams-Moulton method. [7 Marks]

QUESTION B3 [*20 Marks*]

B3 (a) Consider the differential equation

$$-u''(x) + cu'(x) = f(x)$$

where c is a constant and $f(x)$ is a known function of x.

i. Derive the finite difference scheme for solving the above differential equation using central difference quotients for all derivatives. [4 Marks]

ii. Find the local truncation error of the finite difference scheme obtained in (a) and establish if the scheme is consistent [6 Marks]

(b) Consider the heat equation $u_t = u_{xx}$

i. Derive the finite difference scheme for solving the heat equation using the backward difference in time and central difference scheme in space (BTCS). [3 Marks]

ii. Use the von-Neumann analysis analysis to prove that the BTCS scheme for solving the heat equation is unconditionally stable. [7 Marks]

QUESTION B4 [*20 Marks*]

B4 Consider the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} + u &= xy, \quad 0 \le x \le 3,\ 0 \le y \le 1, \\ u(x,0) &= x,\ u(x,1) = 1,\ \ 0 \le x \le 3, \\ u(0,y) &= y,\ u(3,y) = -1,\ \ 0 \le y \le 1. \end{aligned}$$

Suppose that finite difference is used with step sizes $h = 1$ and $k = \frac{1}{2}$, in the x and y-direction, respectively, and the notation $u(x_i, y_j) \approx u_{i,j}$

(a) Indicate the points that represent $u\left(1, \frac{1}{2}\right)$ and $u\left(2, \frac{1}{2}\right)$ on the finite difference grid. [3 Marks]

(b) Compute all the boundary conditions [4 Marks]

(c) Derive the finite difference scheme [3 Marks]

(d) Apply the boundary conditions in the finite difference scheme derived above and solve the resulting system of equations to obtain the approximate solutions for $u\left(1, \frac{1}{2}\right)$ and $u\left(2, \frac{1}{2}\right)$. [10 Marks]

QUESTION B5 [*20 Marks*]

B5 (a) Consider the data in the following table

x	1	2	4	5
y	2	3	5	6

We want to construct the least squares approximation of the form $y = be^{ax}$. Instead of minimising the least squares error associated with $y = b^{ax}$, the problem can be converted to that of minimising the least squares error of the logarithm

$$\ln y = \ln b + ax$$

where a and $\ln b$ are considered to be unknowns. Show that the normal equations are

$$\begin{bmatrix} 4 & \sum_{i=1}^{4} x_i \\ \sum_{i=1}^{4} x_i & \sum_{i=1}^{4} x_i^2 \end{bmatrix} \begin{bmatrix} \ln b \\ a \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} \ln y_i \\ \sum_{i=1}^{4} x_i \ln y_i \end{bmatrix}$$

[10 Marks]

(b) The Legendre polynomials are orthogonal polynomials with respect to the weight function $w(x) = 1$ on the interval $[-1, 1]$.

i. Given that the first Legendre polynomial is $\phi_0(x) = 1$, use the Gram-Schmidt process to find $\phi_1(x)$ and $\phi_2(x)$. [5 Marks]

ii. Prove that the linear least-squares approximation to $f(x) = e^x$ using the Legendre polynomials is

$$P_1(x) = \frac{1}{2}\left[e - \frac{1}{e}\right] + \frac{3}{e}x$$

[5 Marks]

QUESTION B6 [*20 Marks*]

B6 (a) Use the Taylor method of order 2 to solve the initial value problem

$$y' = t^2y^2, \quad y(0) = 2$$

and estimate $y(0.2)$ using $h = 0.1$. [10 Marks]

(b) Consider the initial value problem

$$\begin{aligned} x' &= -x - 3y + 5z, & x(0) &= 0 \\ y' &= -x + y + 4z, & y(0) &= -2 \\ z' &= 5x - y + 4z, & z(0) &= -1 \end{aligned}$$

Use the Euler's method with $h = 0.1$ to approximate the values of $x(0.2)$, $y(0.2)$ and $z(0.2)$. [10 Marks]

END OF EXAMINATION PAPER