University of Eswatini

January 2019 Re-Sit/Supplementary Examination

BSc IV, B.Ed IV, BASS IV

Title of Paper : Numerical Analysis II

Course Number : MAT411/M411

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Show that the differential equation

$$y' = y\cos(t), \quad 0 \le t \le 1, \quad y(0) = 1$$

has a unique solution [5 Marks]

(b) List all the conditions that must be satisfied for an initial value problem to be well-posed. [4 Marks]

(c) Derive an *explicit* finite difference scheme for the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - 1$$

[6 Marks]

(d) Find the equation of a parabola of the form $y = ax^2 + b$ that best represents the following data using the method of least squares.

x	-1	0	1
y	3	1	2

[7 Marks]

(e) Consider the following ordinary differential equation

$$\frac{dy}{dt} = yt - t^2, \quad 0 \le t \le 1.2, \quad y(0) = 1$$

Solve the problem using the improved Euler's method with $h = 0.6$. [5 Marks]

(f) Use the method of undetermined coefficients to derive the two-step Adams-Bashforth multi-step method [7 Marks]

(g) Discuss the consistency, zero-stability and convergence of the linear multi-step method

$$y_{n+2} = 2y_n - y_{n+1} + \frac{h}{2}[5f_{n+1} + f_n]$$

[6 Marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Derive the recurrence formula

$$T_0(x) = 1,\ \ T_1(x) = 1,\ \ T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

where T_n are Chebyshev polynomials of order n
defined by [5 Marks]

$$T_n(x) = \cos(n \arccos(x)),\ \ \text{for each}\ \ n \geq 0\ \ \text{with}\ \ x \in [-1, 1]$$

(b) Show that the general continuous least squares trigonometric polynomial $S_n(x)$ for

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 \leq x \leq \pi \end{cases}$$

is

$$S_n(x) = \frac{2}{\pi} \sum_{k=1}^{n} \left(\frac{1 - (-1)^k}{k} \right) \sin kx$$

[15 Marks]

QUESTION B3 [*20 Marks*]

B3 Consider the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} =& 0, \quad 0 \leq x \leq 2,\ 0 \leq y \leq 3, \\ u(x, 0) =& x/2,\ u(x, 3) = 1,\ \ 0 \leq x \leq 2, \\ u(0, y) =& y/3,\ u(2, y) = 1,\ \ 0 \leq y \leq 3. \end{aligned}$$

Use finite differences on a uniform grid, with $h = k = 1$,
to approximate both $u(1, 1)$ and $u(1, 2)$. [20 Marks]

QUESTION B4 [*20 Marks*]

B4 (a) Use the Runge-Kutta method of order 4 with $h = 0.1$ to solve the given differential equation

$$y' = y - 4y^2, \quad y(0) = -1$$

on $0 \leq t \leq 0.2$ and compare the approximate solution against the exact solution $y(t) = \dfrac{e^t}{4e^t - 5}$. [10 Marks]

(b) Use the Gram-Schmidt procedure to calculate $L_1(x)$ and $L_2(x)$ where $\{L_0(x), L_1(x), L_2(x)\}$ is an orthogonal set of polynomials on $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ and $L_0(x) = 1$ [10 Marks]

QUESTION B5 [*20 Marks*]

B5 Consider the standard initial value problem

$$y' = f(t, y), \quad y(0) = y_0$$

we would like to construct a numerical method from the quadratic interpolant $P_2(t)$, of f at the equally spaced nodes t_{n-1}, t_n and t_{n+1}.

(a) Write down the Newton form of P_2 in forward difference form. [4 Marks]

(b) By integrating between x_n and x_{n+1}, derive the implicit method

$$y_{n+1} = y_{n-1} + \frac{h}{3}\{f_{n-1} + 4f_n + f_{n+1}\}$$

[8 Marks]

(c) Prove that this method is of order 4, and find the leading term in the local truncation error. [8 Marks]

QUESTION B6 [*20 Marks*]

B6 (a) Use finite differences with step size $h = 1$ and central difference approximation on all derivatives to approximate the solution of

$$y'' + y' + xy = 0, \quad y(0) = 0, \quad y(3) = 1$$

[10 Marks]

(b) Find the linear least squares approximation of

$$f(x) = \frac{4}{x+1}$$

in the interval $[0, 1]$. [10 Marks]

END OF EXAMINATION PAPER