UNIVERSITY OF SWAZILAND

MAIN EXAMINATION, 2018/2019

BASS IV, B.Ed (Sec.) IV, B.Sc IV

Title of Paper : COMPUTATIONAL METHODS

Course Number : MAT415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Consider the system $\dot{x} = 4x - y, \dot{y} = 2x + y$.

i. Write the system as $\dot{\mathbf{x}} = A\mathbf{x}$. [1 marks]

ii. Write the matrix A in **LaTeX** form, stating all the commands including the preamble, the packages you would require and the environment in which you would write it. [8 marks]

iii. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, and compute the eigenvalues and eigenvectors of A. [3 marks]

iv. Determine the general solution of the system. [1 marks]

v. Classify the fixed point at the origin. [1 marks]

vi. Write a code or pseudocode in **Mathematica** for solving the system using **NDSolve** subject to $(x_0, y_0) = (3, 4)$. [6 marks]

(b) Precision and clarity in scientific writing is highly imperative. There is no need for complexity. State a substitute for each of the following words to ensure that they are precise and clear. Leave the word as it is in case you think it has no simpler substitute.
i. *Facilitate* ii. *Proximity* iii. *Erroneous* iv. *Disseminate* v. *Commence*. [5 marks]

(c) Consider the Lorenz equations

$$\begin{aligned}
\dot{x} &= -\sigma x + \sigma y \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz,
\end{aligned}$$

where σ, r and b are positive parameters.

i. Find all the fixed points and show that the origin is the only steady-state when $0 < r < 1$. [2 marks]

ii. For the specific values $\sigma = 10$, $b = 8/3$ and $r = 7$, classify the equilibrium point at the origin as either stable or unstable. repeller. [5 marks]

(d) If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of velocity V, then the body's velocity t seconds into the fall satisfies the equation

$$m\frac{dV}{dt} = mg - kV^2, \;\; k > 0, \; g > 0$$

where k is a constant that depends on the body's aerodynamic properties and the density of the air. Assuming that the fall is too short to be affected by changes in the air's density.

i. Draw a phase line for the equation. [3 marks]

ii. Sketch a typical velocity curve. [3 marks]

(e) Sketch a plausible phase portait and classify the fixed point of the linear system $\dot{x} = y$, $\dot{y} = -2x - 3y$. [5 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Define the following terms as used in scientific writing

i. *Hypothesis* [2 marks]

ii. *Theorem* [2 marks]

iii. *Thesis* [2 marks]

iv. *Plagiarism* [2 marks]

(b) Consider the system of differential equations

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{K}\right) - \beta xy \quad (1)$$

$$\frac{dy}{dt} = \beta xy - \gamma y \quad (2)$$

i. Without stating the preamble, write the system in **LaTeX** form. [4 marks]

ii. Given that $\alpha = 0.2$, $\beta = 0.1$, $\gamma = 0.08$, $K = 1000$, $x(0) = 500$ and $y(0) = 100$, write a **MATLAB** or **OCTAVE** script for numerically solving the above system of equations using Euler's method on the time domain; $t \in [0, 10]$. [8 marks]

QUESTION B3 [*20 Marks*]

B3 (a) Consider the following two predator-prey systems of differential equations:

System A.

$$\frac{dx}{dt} = 10x\left(1 - \frac{x}{10} - 20xy\right)$$

$$\frac{dy}{dt} = -5xy + \frac{xy}{20}$$

System B.

$$\frac{dx}{dt} = 0.3x - \frac{xy}{100}$$

$$\frac{dy}{dt} = 15y\left(1 - \frac{y}{15} + 25xy\right)$$

In one of these systems, the prey are very large animals and the predators are very small animals, such as elephants and mosquitos. Thus, it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey, such as whales and krill. Determine which system is which and provide a justification for your answer. [10 marks]

(b) The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition.

i. Discuss the reasonableness of the following model for the growth rate of the deer population as a function of time

$$\frac{dP}{dt} = rP(M-P)(P-m)$$

where P is the population of the deer and r is a positive constant of proportionality (HINT: Use a phase line). [6 marks]

ii. Show that if $P > M$ for all t, then $\lim_{t\to\infty} P(t) = M$. [2 marks]

iii. What happens if $P < m$ for all t? [2 marks]

QUESTION B4 [*20 Marks*]

B4 (a) Consider a model of blood cholesterol levels based on the fact that cholesterol is manufactured by the body for use in the construction of cell walls and is observed from foods containing cholesterol. Let $C(t)$ be the amount (in milligrams per decilitre) of the cholesterol in the blood of a particular person at a time t in days). Then

$$\frac{dC}{dt} = k_1(N-C) + k_2E$$

N is the persons natural cholesterol level, k_1 is a production parameter, E is the daily rate at which cholesterol is eaten and k_2 is the absorption parameter. Suppose that $N = 200$, $k_1 = 0.1$, $k2 = 0.1$, $E = 400$, and $C(0) = 150$. What will the persons cholesterol level be after 2 days on this diet? [8 marks]

(b) The atmospheric pressure, P, in kPa exponentially decreases with increasing height above sea level, h. The pressure can be modelled by the function

$$P(h) = 101 \times \left(\frac{25}{22}\right)^{-h}$$

where h is the height above sea level in kilometres.

i. What is the exact atmospheric pressure at sea level? [2 marks]

ii. Mount Szko has a height of 2228 metres above sea level at the top. Calculate the atmospheric pressure at the top of Mount Szko. [2 marks]

iii. Calculate the height when the atmospheric pressure is 10 kPa. [3 marks]

(c) A ball is fired from the top of a tower. The height, h, in metres of the ball above the ground is modelled by the function

$$h(t) = -2t^2 + 20t + 8, \ t \geq 0$$

where t is the time is seconds from the moment the ball is fired.

i. Calculate the maximum height reached by the ball. [3 marks]

ii. Determine the height of the tower. [2 marks]

QUESTION B5 [*20 Marks*]

B5 Consider the dynamical sytem

$$\dot{x} = x^2 - y - 1$$
$$\dot{y} = (x - 2)y$$

(a) Determine the nullclines and fixed points of the system. [5 marks]

(b) By using the Jacobian matrix of the system, determine the linear stability of all the fixed points. [10 marks]

(c) On the same axes, draw all the nullclines of the system and sketch its phase portrait. [5 marks]

QUESTION B6 [*20 Marks*]

B6 The population dynamics of a predator, P , and its prey, N , are given by the following ordinary differential equations

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - f(N)P$$
$$\frac{dP}{dt} = f(n)P - dP$$

where r, K and d are positive constants and $f(N)$ is a strictly increasing function of N.

(a) Give ecological meanings of the parameters r, K and d. [4 marks]

(b) Let $f(N) = bN$ where b is a positive constant. Sketch phase portraits for this system for when (i) $K < d/b$ and (ii) $K > d/b$. Each should clearly show the nullclines, qualitative directions of flow and two sample trajectories. Hence infer the long-term behaviour in each case. [10 marks]

(c) Now let $f(N) = bN/(N + c)$ where b and c are positive constants. Sketch this function and explain why it might be more realistic than the function in part (b) above. [2 marks]

(d) For $f(N) = bN/(N + c)$, show that there is a coexistence equilibrium at

$$N^* = \frac{cd}{b - d}, \quad P^* = \frac{r}{b}\left(1 - \frac{N^*}{K}\right)(N^* + c).$$

[4 marks]

END OF EXAMINATION