UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2018/2019

BASS IV, B.Ed (Sec.) IV, B.Sc. IV, B.Eng. III

Title of Paper : Partial Differential Equations

Course Number : MAT416/M415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

a) Classify each of the following PDEs according to order, linearity and homogeneity

i) $\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\left(x^9 \frac{\partial^3 \phi}{\partial x^3}\right) = \phi$ [3]

ii) $\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\left(\left(\frac{\partial^2 \phi}{\partial x^2}\right)\left(\frac{\partial^3 \phi}{\partial x^3}\right)\right) - 2x = 0$ [3]

b) Find the solution of [7]

$$xu_x - 4u = x^3, \quad u(1, y) = y^2.$$

c) Consider the partial differential equation (k and r are real constants) [7]

$$ru_{xx} - k^2 r u_{yy} - 2k^2(u_x + u_y) + e^x = 1.$$

Determine the characteristic curves $\xi(x, y)$ and $\eta(x, y)$ of the partial differential equation.

d) Solve the initial value problem [7]

$$u_t - u = e^t, \quad u(x, 0) = e^{-x}, \quad x > 0, \quad t > 0,$$

by using Laplace transforms.

e) By eliminating the arbitrary functions Φ and Ω, find the PDE satisfied by [5]

$$v(r, s) = \Phi(2r + 5s) + \Omega(2r - 5s).$$

f) Consider the heat equation

$$\phi_t = 2\phi_{xx}, \quad 0 < x < \xi, \quad t \geq 0,$$
$$\phi(x, 0) = e^{-x}, \quad 0 \leq x \leq \xi,$$
$$\phi_x(0, t) = \phi(\xi, t) = 0.$$

Write down the ordinary boundary value problem for $X(x)$ that must be solved in order to obtain the solution of the heat equation using the method of separation of variables. Assume that we seek a solution of the form $\phi(x, t) = X(x)T(t)$. [3]

g) The ODE

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2 \frac{dR}{d\rho}\right) = n(n + 1),$$

where n is an arbitrary real constant, arises in the solution of PDEs in spherical domains. Find the general solution of the ODE. [5]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

a) Find the general solution of [8]

$$(y+u)u_x + yu_y = x - y,$$

using the method of characteristics.

b) Consider the PDE

$$u_t - 4u_x = 8xu.$$

Solve the problem using

i) the method of characteristics [4]

ii) the method of separation of variables [8]

QUESTION B3 [*20 Marks*]

a) Consider the partial differential equation

$$u_{yy} + 5u_{xy} + 4u_{xx} + u_x + u_y = 0.$$

(i) Determine whether the given partial differential equation is hyperbolic, parabolic or elliptic. [2]

(ii) Express the given partial differential equation in canonical form. [8]

(iii) Find the general solution of the given partial differential equation. [6]

b) Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$: [4]

$$u_{tt} - 16u_{xx} = 0, \quad u(x,0) - \sin(x) = 0, \quad u_t(x,0) = 4.$$

Determine $u(x,t)$.

QUESTION B4 [*20 Marks*]

Solve the Dirichlet problem of a circle [20]

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 0, \quad 0 \le r \le 1, \quad u(1,\phi) = \frac{\phi}{2}.$$

QUESTION B5 [*20 Marks*]

Consider the radioactive decay problem given by [20]

$$u_t = u_{xx} + 4e^{-x}, \quad 0 < x < \pi, \quad t > 0,$$
$$u(x,0) = \sin(x), \quad 0 \le x \le \pi,$$
$$u(0,t) = 0, \quad u(\pi,t) = 0,$$

Determine $u(x,t)$ using the method of separation of variables.

QUESTION B6 [*20 Marks*]

a) Find the corresponding ordinary differential equation after taking the Laplace transform of the given partial differential equation with respect to t [4]

$$\begin{aligned} 4u_{xx} - u_t &= 0, \quad 0 < x < 4, \quad t > 0, \\ u(x,0) &= 2x, \quad 0 \leq x \leq 4, \\ u(0,t) &= \pi, \quad u_x(4,t) = t^2. \end{aligned}$$

b) Use Laplace transforms to find a solution of [16]

$$\begin{aligned} u_{xx} - u_{tt} &= -\cos\left(\frac{\pi}{2}x\right), \quad 1 \leq x \leq 3, \quad t \geq 0, \\ u(x,0) &= 0, \quad 1 \leq x \leq 3, \\ u_t(x,0) &= 0, \\ u(1,t) &= 0, \quad u(3,t) = 0. \end{aligned}$$

END OF EXAMINATION PAPER

$f(t)$	$f(t) = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$t^n \ (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
e^{at}	$\frac{1}{s-a}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$

$f(t)$	$f(t) = F(s)$
$\frac{ae^{at}-be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$
$e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$
$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$
$t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$
$\frac{\sin at}{t}$	$\arctan \frac{a}{s}$
$\frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$