UNIVERSITY OF ESWATINI

RESIT/SUPPLEMENTARY EXAMINATION, 2018/2019

BASS IV, B.Ed (Sec.) IV, B.Sc. IV, B.Eng. III

Title of Paper : Partial Differential Equations

Course Number : MAT416/M415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

a) Determine the order of the partial differential equation satisfied by [3]

$$\rho(r,s) = \Omega(r-s) + \Psi(5r-s) + \Phi(5r+s),$$

where Ω, Ψ and Φ are arbitrary functions.

b) Given that the arbitrary function $F(x,y)$ is differentiable, show that [5]

$$u(x,y) = xe^x + xF(y-x^2)$$

satisfies

$$xu_x + 2x^2u_y - u = x^2e^x.$$

c) Suppose that the temperature distribution in a rod of length $5m$ is given by $T(x,t)$. Assuming that one end is kept at zero temperature and the other end is insulated such that there is no heat flow, write down a model that could be used to determine the temperature distribution $T(x,t)$, provided that the initial temperature distribution is given by e^{-x}. [5]

d) Derive Parseval's identity theorem for the summability of the Fourier series coefficients of a function. [6]

e) Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$:

$$f_{tt} - f_{xx} = 0, \qquad f(x,0) = x-1, \qquad f_t(x,0) = 2x.$$

Determine $f(1,1)$. [7]

f) Consider the initial-value problem Solve the problem using

i) Solve

$$-4u_x + u_t = 8xu,$$

using the method of characteristics [7]

ii) Solve

$$xu_x + u_t = x, \quad u(x,0) = u(0,t) = 0,$$

using the method Laplace transforms [7]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

a) Consider the partial differential equation (PDE)

$$u_{xx} - 2\sin(x)u_{xy} - \cos^2(x)u_{yy} + e^x = 0.$$

(i) Classify the PDE by stating its order, linearity, and homogeneity. [3]

(ii) Determine whether the PDE is hyperbolic, parabolic or elliptic. [2]

(iii) Express the PDE in canonical form. [7]

b) Find the general solution of [8]

$$(x-y)(y^2u_x - x^2u_y) = (x^2+y^2)u,$$

using the method of characteristics.

QUESTION B3 [*20 Marks*]

Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$:

$$\rho_{tt} = v^2\rho_{xx},$$
$$\rho(x,0) = \phi(x),$$
$$\rho_t(x,0) = \psi(x),$$

where v is a constant. Show that the solution of the wave equation is given by: [20]

$$\rho(x,t) = \frac{1}{2}\left(\phi(x+vt) + \phi(x-vt) + \frac{1}{v}\int_{x-vt}^{x+vt}\psi(\gamma)d\gamma\right)$$

QUESTION B4 [*20 Marks*]

(a) Use Laplace transforms to find a solution [15]

$$u_{xx} - u_t = \sin(\pi x), \quad 0 \le x \le 3, \quad t > 0,$$
$$u(x,0) = 0, \quad 0 \le x \le 3,$$
$$u(0,t) = 0, \quad u(3,t) = 0$$

(b) Using the fact that the Laplace transform of $u(x,t)$ with respect to the variable t is given by

$$\mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-st}u(x,t)dt \equiv U(x,s),$$

Show that $\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x,s) - u(x,0)$ [5]

QUESTION B5 [*20 Marks*]

Consider the following wave equation given by

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \quad 0 < x < \pi, \quad t > 0, \\ u(x,0) &= 0, \quad 0 \leq x \leq \pi, \\ u_t(x,0) &= 2\sin(x), \\ u(0,t) &= 0, \\ u(\pi,t) &= 0 \end{aligned}$$

Find $u(x,t)$ using the method of separation of variables. [20]

QUESTION B6 [*20 Marks*]

Consider the Dirichlet problem of a sphere of radius $r = a$.

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) = 0, \quad 0 \leq r \leq a.$$
$$u(a,\phi) = f(\phi), \quad 0 \leq \phi \leq \pi.$$

Use the method of separation of variables to determine $u(r,\phi)$. [20]

END OF EXAMINATION PAPER

$f(t)$	$f(t) = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$t^n \ (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
e^{at}	$\frac{1}{s-a}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$

$f(t)$	$f(t) = F(s)$
$\frac{ae^{at}-be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$
$e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$
$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$
$t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$
$\frac{\sin at}{t}$	$\arctan\frac{a}{s}$
$\frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$