UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2018/2019

BASS, B.Ed (Sec.), B.Sc.

Title of Paper : Optimisation Theory

Course Number : MAT418

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, A2, B2, ..., B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Some formulas are given on the last page.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*20 Marks*]

(a) Give precise definitions of the following.

i. Convex set S in $\mathbb{R}^n$. (2)

ii. Convex function from a convex set $S \subseteq \mathbb{R}^n$ to $\mathbb{R}$. (2)

iii. Concave function from a convex set $S \subseteq \mathbb{R}^n$ to $\mathbb{R}$. (2)

(b) Show that $f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$ is a convex function on $\mathbb{R}^2$. (4)

(c) Show that $f(x_1, x_2) = -x_1^2 - x_1x_2 - 2x_2^2$ is a concave function on $\mathbb{R}^2$. (4)

(d) Find the optimal solution to

$$\begin{array}{ll} \max & x^3 - 3x^2 + 3x - 1 \\ \text{s.t.} & -2 \leq x \leq 4 \end{array}$$

(6)

QUESTION A2 [*20 Marks*]

(a) Use the graphical method to solve the following LP. State which constraint is binding and which is non-binding (if any).

$$\begin{array}{lrcrcl} \max\ z = & x_1 & + & x_2 & & \\ \text{s.t.} & 2x_1 & + & 3x_2 & \leq & 18 \\ & 2x_1 & + & x_2 & \leq & 12 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

(10)

(b) Consider the following LP.

$$\begin{array}{lrcrcl} \max\ z = & 3x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \geq & 3 \\ & 2x_1 & + & x_2 & \leq & 4 \\ & x_1 & + & x_2 & = & 4 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

i. Write down the initial simplex tableau for the "Big-M" method. (5)

ii. Perform one step of the "Big-M" method to find a new bfs. Is the new bfs optimal? (5)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [20 *Marks*]

(a) Find all local extrema and saddle points of the function

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1^2 x_2 + 4.$$

(10)

(b) Use the method of steepest ascent to approximate the solution to

$$\begin{aligned} \max \ z &= -(x_1 - 3)^2 - (x_2 - 2)^2 \\ \text{s.t.} \quad & (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Start at the point $(1, 1)$. (10)

QUESTION B4 [20 *Marks*]

(a) A company is planning to spend E10,000 on advertising. It costs E3,000 per minute to advertise on TV and E1,000 per minute to advertise on radio. If the company buys x minutes of TV adverts and y minutes of radio adverts, its revenue (in thousands of emalangeni) is given by

$$R(x, y) = -2x^2 - y^2 + xy + 8x + 3y.$$

Use Lagrange multipliers to determine the values of x and y that will maximise the company's revenue. (10)

(b) The Douglas-Cobb model says that when a company invests L units of labour and K units of capital, the production level P is given by

$$P = bL^{\alpha}K^{1-\alpha}$$

where $b > 0$ and $0 < \alpha < 1$ are constants. Suppose that the cost per unit labour is m emalangeni and the cost per unit capital is n emalangeni and that the company has a budget of B emalangeni to spend on total labour and capital. Show that maximum production occurs when

$$L = \frac{\alpha B}{m} \quad \text{and} \quad K = \frac{(1-\alpha)B}{n}.$$

(10)

QUESTION B5 [20 *Marks*]

(a) Consider the following LP.

$$\begin{array}{llrcrcrcr} \max & z = & 2x_1 & - & 3x_2 & + & x_3 & & \\ \text{s.t.} & & 6x_1 & + & 8x_2 & + & x_3 & \leq & 100 \\ & & 4x_1 & + & 3x_2 & - & 2x_3 & \leq & 90 \\ & & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

After adding slack variables s_1 and s_2, and solving using the simplex algorithm, the basic variables in the LP's optimal solution are $BV = \{x_3, s_2\}$ (in that order).

Construct the LP's optimal tableau using formulas. (10)

(b) Consider the following LP.

$$\begin{array}{llrcrcr} \max & z = & 3x_1 & + & 2x_2 & & \\ \text{s.t.} & & 2x_1 & + & x_2 & \leq & 100 \\ & & x_1 & + & x_2 & \leq & 80 \\ & & x_1 & & & \leq & 40 \\ & & & & x_1, x_2 & \geq & 0 \end{array}$$

After adding slack variables s_1, s_2, s_3 and solving using the simplex algorithm, the optimal tableau is found to be

z	x_1	x_2	s_1	s_2	s_3	rhs
1	0	0	1	1	0	180
0	1	0	1	−1	0	20
0	0	1	−1	2	0	60
0	0	0	−1	1	1	20

i. Show that the current basis remains optimal for $1.5 \leq c_2 \leq 3$. (5)

ii. Show that the current basis remains optimal for $80 \leq b_1 \leq 120$. (5)

Hint:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

QUESTION B6 [20 *Marks*]

Use the Kuhn-Tucker conditions to find the optimal solution to the following problem.

$$\begin{aligned} \max\ z &= x_1(30 - x_1) + x_2(50 - 2x_2) - 3x_1 - 5x_2 - 10x_3 \\ \text{s.t.} \quad & x_1 + x_2 - x_3 \leq 0 \\ & x_3 \leq 18 \end{aligned}$$

QUESTION B7 [20 *Marks*]

Consider the following LP.

$$\begin{aligned} \max\ z &= 4x_1 + x_2 + 2x_3 \\ \text{s.t.} \quad & 8x_1 + 3x_2 + x_3 \leq 2 \\ & 6x_1 + x_2 + x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(a) Find the dual of the LP. (4)

(b) Use the graphical method to solve the dual of the LP. (8)

(c) Use complementary slackness to solve the primal LP. (8)

END OF EXAMINATION PAPER

USEFUL FORMULAS

$\bar{c}_j = \mathbf{c}_{BV}B^{-1}\mathbf{a}_j - c_j, \quad \bar{\mathbf{a}}_j = B^{-1}\mathbf{a}_j, \quad \bar{\mathbf{b}} = B^{-1}\mathbf{b}$

$\bar{z} = \mathbf{c}_{BV}B^{-1}\mathbf{b}, \quad \bar{c}_{s_i} = i\text{-th element of } \mathbf{c}_{BV}B^{-1}.$