UNIVERSITY OF ESWATINI

RE-SIT EXAMINATION, 2018/2019

BASS, B.Ed (Sec.), B.Sc.

Title of Paper : Optimisation Theory

Course Number : MAT418

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, A2, B3, ..., B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Some formulas are given on the last page.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*20 Marks*]

(a) Determine whether the given function is a convex function, concave function or neither on the given set S. Explain.

i. $f(x_1, x_2) = x_1^2 - 3x_1x_2 + 2x_2$ on $S = \mathbb{R}^2$. (5)

ii. $f(x_1, x_2) = x_1^2 + x_2^2$ on $S = \mathbb{R}^2$. (5)

iii. $f(x) = x^3$ on $S = (-\infty, 0]$. (4)

(b) Let $f(x) = \begin{cases} 2 - (x-a)^2, & 0 \le x < 3 \\ -3 + (x-4)^2, & 3 \le x \le 6. \end{cases}$

Find the optimal solution to

$$\begin{array}{ll} \max & z = f(x) \\ \text{s.t.} & -2 \le x \le 4 \end{array}$$

(6)

QUESTION A2 [*20 Marks*]

(a) Use the graphical method to solve the following LP. State which constraint is binding and which is non-binding (if any).

$$\begin{array}{lrcrcl} \min\ z = & -3x_1 & - & x_2 & & \\ \text{s.t.} & x_1 & - & x_2 & \le & 1 \\ & x_1 & + & x_2 & \le & 2 \\ & & & x_1, x_2 & \ge & 0 \end{array}$$

(10)

(b) Consider the following LP.

$$\begin{array}{lrcrcl} \min\ z = & 3x_1 & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & \ge & 6 \\ & 3x_1 & + & 2x_2 & = & 4 \\ & & & x_1, x_2 & \ge & 0 \end{array}$$

i. Write down the initial simplex tableau for the "Big-M" method. (4)

iii. Perform one step of the "Big-M" method to find a new bfs. Is the new bfs optimal? (6)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [20 *Marks*]

(a) Find all local extrema and saddle points of the function

$$f(x_1, x_2) = x_1^4 + x_2^4 - 4x_1x_2 + 1.$$

(10)

(b) Use the method of steepest ascent to approximate the solution to

$$\begin{array}{lrl} \max & z = & -(x_1-2)^2 - x_1 - x_2^2 \\ \text{s.t.} & & (x_1, x_2) \in \mathbb{R}^2. \end{array}$$

Start at the point $\left(\frac{5}{2}, \frac{3}{2}\right)$. (10)

QUESTION B4 [20 *Marks*]

(a) It costs E20 to purchase 1 hour of labour and E10 to purchase a unit of capital. If L hours of labour and K units of capital are available, then $L^{2/3}K^{1/3}$ machines can be produced. If E100 is available to purchase labour and capital, what is the maximum number of machines that can be produced?

(10)

(b) Find the optimal solution to the following problem.

$$\begin{array}{lrl} \max & z = & x_1^2 + 2x_2^2 \\ \text{s.t.} & & x_1^2 + x_2^2 = 1 \\ & & (x_1, x_2, x_3) \in \mathbb{R}^3 \end{array}$$

(10)

QUESTION B5 [*20 Marks*]

(a) Consider the following LP.

$$\begin{array}{llcl} \max\ z & = & 3x_1 + 2x_2 & \\ \text{s.t.} & & 2x_1 + x_2 & \leq 100 \\ & & x_1 + x_2 & \leq 80 \\ & & x_1 & \leq 40 \\ & & x_1, x_2 & \geq 0 \end{array}$$

After adding slack variables s_1, s_2, s_3, and solving using the simplex algorithm, the basic variables in the LP's optimal solution are $BV = \{x_1, x_2, s_3\}$ (in that order).

Construct the LP's optimal tableau using formulas.

Hint:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

(10)

(b) Consider the following LP.

$$\begin{array}{llcl} \max\ z & = & 3x_1 + 7x_2 + 5x_3 & \\ \text{s.t} & & x_1 + x_2 + x_3 & \leq 50 \\ & & 2x_1 + 3x_2 + x_3 & \leq 100 \\ & & x_1, x_2, x_3 & \geq 0 \end{array}$$

After adding slack variables s_1 and s_2, the optimal tableau is found to be

z	x_1	x_2	x_3	s_1	s_2	rhs	BV
1	3	0	0	4	1	300	$z = 300$
0	$\frac{1}{2}$	0	1	$\frac{3}{2}$	$-\frac{1}{2}$	25	$x_3 = 25$
0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	25	$x_2 = 25$

Use the optimal tableau to answer the following questions:

i. For what values of c_1 (the objective function coefficient of x_1) does the current basis remain optimal? (5)

i. For what values of b_1 (the right-hand-side of constraint 1) would the current basis remain optimal? (5)

QUESTION B6 [*20 Marks*]

(a) Use the Kuhn-Tucker conditions to find the optimal solution to the following problem.

$$\begin{array}{lrcl} \max\ z & = x_1 - x_2 & & \\ \text{s.t.} & x_1^2 + x_2^2 & \leq & 1 \\ & x_1, x_2 & \in & \mathbb{R} \end{array}$$

(10)

(b) Perform *one* iteration of the feasible directions method on the following problem.

$$\begin{array}{ll} \max & z = 2xy + 4x + 6y - 2x^2 - 2y^2 \\ \text{s.t.} & x + y \leq 2 \\ & x, y \geq 0 \end{array}$$

Begin at the point $(0, 0)$. (10)

QUESTION B7 [*20 Marks*]

Consider the following LP.

$$\begin{array}{lrcrcrcl} \max\ z = & 5x_1 & + & 3x_2 & + & x_3 & & \\ \text{s.t.} & 2x_1 & + & x_2 & + & x_3 & \leq & 6 \\ & x_1 & + & 2x_2 & + & x_3 & \leq & 7 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

(a) Find the dual of the LP. (4)

(b) Use the graphical method to solve the dual of the LP. (8)

(c) Use complementary slackness to solve the primal LP. (8)

END OF EXAMINATION PAPER

USEFUL FORMULAS

$\bar{c}_j = \mathbf{c}_{BV}B^{-1}\mathbf{a}_j - c_j, \quad \bar{\mathbf{a}}_j = B^{-1}\mathbf{a}_j, \quad \bar{\mathbf{b}} = B^{-1}\mathbf{b}$

$\bar{z} = \mathbf{c}_{BV}B^{-1}\mathbf{b}, \quad \bar{c}_{s_i} = i$-th element of $\mathbf{c}_{BV}B^{-1}$.