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University of Eswatini

Main Examination, 2018/2019

BASS IV, B.Ed. (Sec.) IV; B.Sc IV

Title of Paper : Metric Space

Course Number : M431/MAT434

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) i. Define a metric space. [5 marks]

ii. Consider $X = \mathbb{R}$. Define $d : X \times X \to [0, \infty)$ by $d(x, y) = |x^2 - y^2|$. Verify whether or not d is a metric on $\mathbb{R}$. [3 marks]

(b) Let (X, d) be a metric space and $A \subset X$. Define the following terms:

i. an open ball in X. [2 marks]

ii. A is an open set in X. [2 marks]

iii. interior point of A in X. [2 marks]

iv. limit point of A in X. [2 marks]

(c) Write True/False in each of the following questions:

i. Let (X, d) be a complete metric space and $S \subseteq X$. Then A is complete if and only if A is not closed. [2 marks]

ii. Let $f : [a, b] \to \mathbb{R}$ be continuous. f is uniformly continuous on $[a, b]$. [2 marks]

(d) Let (X, d) be a metric space.

i. When do we say that (X, d) is a complete metric space. [3 marks]

ii. Define a contraction mapping on (X, d). [3 marks]

iii. State Banach Contraction Mapping Principle? [4 marks]

iv. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = \left(\frac{y}{2}, \frac{x}{3}\right)$. Show that f is a contraction on $\mathbb{R}^2$ (with respect to the Euclidean metric). [5 marks]

(e) i. Find the limit of the sequence

$$x_n := \left(\frac{1}{n^2}, \frac{n}{n+1}\right).$$

[2 marks]

ii. Let $X = \mathbb{R}$ (the real) endowed with the usual metric. Let $E = (0, \infty)$. Show that 0 is a limit point of E. [3 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Let $X = \mathbb{R}^2$, for each $(x_1, x_2), (y_1, y_2) \in X$, define $d_1 : X \times X \to \mathbb{R}$ by $d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Show that d_1 is a metric on X. [11 marks]

(b) Let $X = \mathbb{R}$(the real line) with metric ρ_0 defined by

$$\rho_0(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

for arbitrary $x, y \in \mathbb{R}$. Find the following open balls: [9 marks]
(i) $B_{\frac{1}{2}}(1)$; (ii) $B_2(1)$; (iii) $B_1(5)$.

QUESTION B3 [*20 Marks*]

B3 Let (X, d) be a metric space.

(a) Prove that an arbitrary union of open sets in X is open in X. [7 marks]

(b) Prove that every open ball $B(x, r) \subset X$ is an open set in X. [7 marks]

(c) Let $X = \mathbb{R}^2$ and d_2, d_∞ be metric in $\mathbb{R}^2$.
If $(x_1, y_1) = (-3, 4)$ and $(x_2, y_2) = (20, 2)$. Find

i. $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$; [3 marks]

ii. $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. [3 marks]

QUESTION B4 [*20 Marks*]

B4 (a) Let (X, ρ_X) and (Y, ρ_Y) be any two metric spaces. Define a continuous mapping $f : (X, \rho_X) \to (Y, \rho_Y)$. [5 marks]

(b) Let (X, ρ_X) and (Y, ρ_Y) be any two metric spaces. Let $f : X \to Y$ be defined by $f(x) = y_0$(constant) for all $x \in X$. Prove that f is continuous. [6 marks]
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} x^2 \sin\left(\frac{1}{y}\right) + y^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \text{ and } y \neq 0 \\ 0, & \text{for } x = 0 \text{ and } y = 0 \end{cases}$$

Prove that f is continuous at $(0, 0)$. [9 marks]

QUESTION B5 [*20 Marks*]

B5 (a) Let (X, d) be a metric space and for any $x, y, w, z \in X$.
Prove that

$$\Big|d(x, y) - d(w, z)\Big| \leq d(w, x) + d(z, y).$$

[5 marks]

(b) Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a metric space (X, d). Prove that $\lim_{n\to\infty} d(x_n, y_n)$ exists. [8 marks]

(c) Let $CS(X)$ be a set of Cauchy sequence in a metric space (X, d).
For any $\{x_n\}, \{y_n\}$ in $CS(X)$, define a relation " $\sim$ " to mean that $\{x_n\} \sim \{y_n\}$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$. Prove that $\sim$ is an equivalence relation. [7 marks]

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QUESTION B6 [*20 Marks*]

B6 (a) Let (X, d) be a complete metric space and $E \subseteq X$. Prove that E is complete if and only if E is closed. [10 marks]

(b) Let K be a subset of a metric space X. Under what condition is K compact? [4 marks]

(c) Define a Homeomorphism. [6 marks]

END OF EXAMINATION