# University of ESwatini

## Final Examination, December 2018

## B.A.S.S. , B.Sc, B.Ed

**Title of Paper** : Dynamics II

**Course Number** : M355/MAT455

**Time Allowed** : Three (3) Hours

**Instructions**

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      **Submit solutions to ONLY THREE questions in Section B.**
2. Begin each major question (A1, B2, etc) on a new page.
3. Each question in Section B is worth 20%.
4. Show all your working.
5. Non programmable calculators may be used (unless otherwise stated).
6. Special requirements: None.
7. Indicate your program next to your student ID number.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

# Section A: Answer All Questions

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**A1.**

(a) i. Consider a particle moving on a circle, in the $xy-$plane with radius $R$ and centred at the origin. Is the system holonomic or non-holonomic? Write the equations of the constrains if possible. If the system is holonomic, determine the number of degrees of freedom. [4]

ii. For a certain system the kinetic energy $T$ and potential energy $V$ are given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2), \quad V = V(y),$$

where $x$ and $y$ are generalized coordinates. Write down Lagrange's equations for the system. [7]

iii. When is a mechanical system said to be conservative? [3]

iv. State D'Alembert's Principle. [3]

v. In your own words, give the main differences between Lagrangian and Hamiltonian mechanics. [4]

(b) i. Is the transformation $P = -q, \quad Q = p + 2q^2$, canonical? [5]

ii. Evaluate the Poisson bracket, $[q^2p, \; qp]$. [6]

iii. Differentiate between Essential and Natural boundary conditions. [3]

iv. Show that the differential equation for the functional

$$I = \int_a^b x(y'^2 - y^2)dx$$

is

$$y'' + \frac{1}{x}y + y = 0$$

[5]

## Section B: Answer Three(3) Questions Only

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**B2.** (a) Consider the following equation derived from the D'Alembert's Principle,

$$\sum_{j}^{n} Q_j \delta q_j = \sum_{i}^{N} \sum_{j}^{n} \left[ \frac{d}{dt} \left( m_i \dot{\mathbf{r}}_i \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} \right) - m_i \dot{\mathbf{r}}_i \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \right] \delta q_j \qquad (1)$$

where $Q_j$'s are the generalized forces associated with the generalized coordinates $q_j$'s and $\mathbf{r}_i$'s are the position vectors for the system's particles. Given also that the kinetic energy of a system of $N$ particles is

$$T = \frac{1}{2} \sum_{i}^{N} m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} \sum_{i}^{N} m_i \dot{\mathbf{r}}_i . \dot{\mathbf{r}}_i . \qquad (2)$$

Show that equation (1) reduces to the Lagrange's equations of motion

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j, \quad j = 1, 2, 3, \ldots, n.$$

for conservative systems with holonomic constraints. [6]

(b) The Lagrangian of a system is give by

$$L = \frac{1}{2}(M + m)\dot{x}^2 + \frac{1}{2} m \dot{y}^2 + m \dot{x} \dot{y} \cos\theta - mg(h - y \sin\theta),$$

where $x$ and $y$ are the generalised coordinates, $h$ is a constant.

i. Derive the Lagrange's equations of motion for the system. [8]

ii. Find the canonical momenta $p_x$ and $p_y$ and show that

$$\dot{x} = \frac{p_x - p_y \cos\theta}{M + m \sin^2\theta}, \quad \dot{y} = \frac{(m + M)p_y - m p_x \cos\theta}{m(M + m \sin^2\theta)}.$$

[6]

**B3.** (a) Derive Hamilton's equations when $H$ does not contain time $t$ explicitly. [6]

(b) A Hamiltonian for a system is given by $H = \frac{1}{2}\left(\frac{1}{q^2} + p^2q^4\right)$.

i. Find the corresponding Lagrangian assuming that $p$ is the generalized momentum and $q$ is a generalized coordinate corresponding to $p$. [6]

ii. Determine Hamilton's equations of motion and show that the equation of motion corresponding to $q$ is $q\ddot{q} = 2\dot{q}^2 + q^2$. [8]

**B4.** (a) By any method you choose, show that the following transformation is canonical.

$$Q = \ln\left(\frac{p}{q}\right), \quad P = -\left(\frac{q^2}{2} + 1\right)\frac{p}{q}$$

[10]

(b) For what values of $\alpha$ is the following transformation canonical?

$$P = -\sin\alpha q + \cos\alpha p, \quad Q = \cos\alpha q + \sin\alpha p.$$

[10]

**B5.** (a) Consider a system with Hamiltonian $H = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2)$. Show that $M = p_1p_2 + q_1q_2$ and $L = p_1q_2 - q_1p_2$ are constants of motion by evaluating the Poisson brackets $[M,\ H]$ and $[L,\ H]$. [10]

(b) Find the curve $y(x)$ that minimises the functional

$$I = \int_{-1}^{1}(x^2y'^2 + 12y^2)dx, \quad y(-1) = -1, \quad y(1) = 1.$$

[10]

**B6.** (a) Show that if $F$ has no explicit dependence on $x$ (i.e $\partial F/\partial x = 0$) then $F = F(y, y')$ and the Euler-Lagrange's equation simplifies to the Beltrami-identity which is defined by the equation

$$F - y'\frac{\partial F}{\partial y'} = C.$$

[10]

(b) Show that Euler-Lagrange equation for the functional

$$I = \int_a^b y\sqrt{1 + (y')^2}dx,$$

is given by

$$yy'' = 1 + (y')^2.$$

[10]

END OF EXAMINATION