University of ESwatini

Supplementary/Resit Examination, January 2019

B.A.S.S. , B.Sc, B.Ed

Title of Paper : Dynamics II

Course Number : M355/MAT455

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Begin each major question (A1, B2, etc) on a new page.
3. Each question in Section B is worth 20%.
4. Show all your working.
5. Non programmable calculators may be used (unless otherwise stated).
6. Special requirements: None.
7. Indicate your program next to your student ID number.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A: Answer All Questions

A1.

(a) i. Lagrangian mechanics are not independent of Newton's second law. True or False? Justify. [3]

ii. Give 3 examples of generalised coordinates. [3]

iii. Differentiate, giving examples, between holonomic and non-holonomic constraints. [4]

iv. Find the equations of motion associated with the following Lagrangian function for the indicated generalized coordinates (assume all other parameters are constants).

$$L = \frac{1}{2}(\dot{x_1}^2 + \dot{x_2}^2) - \frac{1}{2}\kappa(x_2 - x_1 - a)^2, \quad (x_1, x_2)$$

[8]

v. Does the system with the Lagrangian in (iv) above have a cyclic coordinate? [2]

(b) i. A one-dimensional harmonic oscillator has Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$. Write down Hamilton's equation and find the general solution. [5]

ii. Is the transformation $Q = e^q, \quad P = p$, canonical? [5]

iii. Evaluate the Poisson bracket, $[q^2p, \ qp]$. [5]

iv. Find the curve $y(x)$ that minimises the functional

$$\int_0^{\pi/2} (y'^2 - y^2)dx, \quad y(0) = 0, \quad y(\pi/2) = 1.$$

[5]

Section B: Answer Three(3) Questions Only

B2. (a) Consider a system of N particles of masses m_i and position vectors $\mathbf{r}_i$. We know the position vectors $\mathbf{r}_i$ are expressed as the functions of n generalized coordinates $q_1, q_2, q_3, \ldots, q_n$ and time t as

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, q_3, \ldots, q_n, t) \tag{1}$$

Use equation (1) to derive the relation,

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{2}$$

called the cancellation of dots property. [8]

(b) The Lagrangian for a certain dynamical system is given by

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + t^2 - 2\dot{r}t\cos\theta + 2rt\dot{\theta}\sin\theta\right) - mg\left(\frac{1}{2}t^2 - r\cos\theta\right) - \frac{k}{2}(r-a)^2,$$

where r and θ are generalized coordinates, t is time and m, g and k are constants. Using the Lagrangian method, show that the equations of motion for the system are given by

$$\ddot{r} - r\dot{\theta}^2 = (g+1)\cos\theta - \frac{k}{m}(r-a)$$

$$\ddot{\theta} + 2\frac{\dot{r}\dot{\theta}}{r} + \frac{g+1}{r}\sin\theta = 0$$

[12]

B3. If the Kinetic energy, T, and Potential energy, V , of a system are given by

$$2T = ml^2(\dot{q_1}^2 + \dot{q_2}^2 \sin^2 q_1), \quad -V = mgl\cos q_1.$$

(a) Find the Hamiltonian, H of the system. [10]

(b) Obtain the Hamilton's equations of motion and deduce that the equation of motion in q_1 is given by

$$\ddot{q_1} = \frac{c^2 \cos q_1}{\sin^3 q_1} - \frac{g}{l}\sin q_1,$$

where $c = p_2/ml^2$. [10]

B4. (a) If the Hamiltonian of a system is given by $H = \frac{1}{\beta}p^\beta$ with β constant, find the corresponding Lagrangian assuming that p is the generalized momentum and q is a generalised coordinate corresponding to p. [8]

(b) Use Poisson brackets to show that the following transformation is canonical.

$$q_1 = \sqrt{2P_1}\sin Q_1 + P_2; \quad p_1 = \frac{1}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right),$$
$$q_2 = \sqrt{2P_1}\cos Q_1 + Q_2, \quad p_2 = -\frac{1}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right).$$

[12]

B5. (a) Consider a system with Hamiltonian $H = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2)$. Show that

$M = p_1p_2 + q_1q_2$ and $L = p_1q_2 - q_1p_2$ are constants of motion by evaluating the

Poisson brackets $[M, H]$ and $[L, H]$. [10]

(b) Find the curve $y(x)$ that minimises the functional

$$I = \int_{-1}^{1}(x^2y'^2 + 12y^2)dx, \quad y(-1) = -1, \quad y(1) = 1.$$

[10]

B6. (a) For what values of the constant parameters α and β is the following transformation canonical?

$$q = \beta P^{\alpha} \sin Q, \quad p = \beta P^{\alpha} \cos Q.$$

[8]

(b) Show that if the integrand F does not depend on y, that is if $F = F(x, y')$, then the Euler-Lagrange equation simplifies to

$$\frac{\partial F}{\partial y'} = C.$$

[4]

(c) Use the result in (b) above to show that the function $y(x)$ that minimizes the functional

$$I = \int_1^2 \frac{\sqrt{1 + y'^2}}{x} dx, \quad y(1) = 0, \quad y(2) = 1,$$

is

$$(y - 2)^2 + x^2 = 5.$$

[8]

END OF EXAMINATION