UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2019/2020

B.A.S.S. II, B.Ed (Sec.) II, B.Sc. II, B.Eng. II

Title of Paper : Linear Algebra

Course Number : MAT221

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [40 Marks]: ANSWER ALL QUESTIONS

QUESTION A1 [40 Marks]

a) Suppose that $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ are vectors in $\mathbb{R}^2$. Determine whether S is linearly dependent or not. [4]

b) Let $A = \begin{bmatrix} 2 & 4 & 4 & 6 \\ 0 & -1 & 1 & 9 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. Find the eigenvalues of A^6. [4]

c) Calculate A^3 using the Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$. [4]

d) Let V_{nn} be the vector space of $n \times n$ matrices. Determine whether the transformation $T(A) = \det(A)$ is linear or not. [4]

e) Suppose that the matrix F is a result of exchanging two rows of matrix G. Given that $|G| = -\pi$, find $|F|$ and $|FG|$. [4]

f) Let $F = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ and $p(x) = 4 - 3x$. Find $p(F)$. [4]

g) Given that $B = \begin{bmatrix} 2 & 4 \\ -1 & -1 \end{bmatrix}$. Find B^{-1} using elementary row operations. [4]

h) Show that if A is an invertible symmetric matrix, then A^{-1} is symmetric. [4]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [20 Marks]

a) Find $|A|$ where [15]

$$A = \begin{bmatrix} 2 & 4 & 4 & 6 \\ -1 & -1 & 1 & 9 \\ -2 & 2 & 1 & 5 \\ -4 & 1 & 2 & 5 \end{bmatrix}.$$

b) Let α be any real nonzero constant and let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ a_{31} & a_{32} & a_{33}. \end{bmatrix}$$

Using the cofactor expansion notation on the second row, show that $|B| = \alpha|A|$. [5]

QUESTION B3 [*20 Marks*]

a) Let $P = \{\mathbf{p}_1, \mathbf{p}_2, \cdots \mathbf{p}_k\}$ be a set of vectors in $\mathbb{R}^n$. Prove that if $k > n$, then P is linearly dependent. [8]

b) Determine whether the vectors $\mathbf{v}_1 = (0, 3, 1, -1)$, $\mathbf{v}_2 = (6, 0, 5, 1)$, $\mathbf{v}_3 = (4, -7, 1, 3)$, are linearly dependent or linearly independent in $\mathbb{R}^4$. If they are linearly dependent, express $\mathbf{v}_3$ as a linear combination of the other vectors. [12]

QUESTION B4 [*20 Marks*]

a) Prove that a square matrix A is invertible if and only if $\sigma = 0$ is not an eigenvalue of A.[8]

b) Find a matrix P that diagonalizes the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 7 & 6 & -1 \end{bmatrix}$ and hence write down an expression in terms of the matrix P that can be used to evaluate A^{10}. [12]

QUESTION B5 [*20 Marks*]

a) Prove that every system of linear equations has no solutions, or exactly one solution or has infinitely many solutions. [10]

b) i) Show that a transpose of a symmetric matrix C is symmetric. [4]

ii) Find A^{-5} given that $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. [6]

QUESTION B6 [*20 Marks*]

a) Solve the system of linear equations [10]

$$3x_1 + x_2 + x_3 + x_4 = 0$$
$$5x_1 - x_2 + x_3 - x_4 = 0.$$

b) Define $T : \mathbb{C}^3 \to \mathbb{C}^2$ by describing the output of the function for a generic input with the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix}$$

Determine whether the transformation is linear or not. [10]

END OF EXAMINATION PAPER