UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2019/2020

BASS, B.Ed (Sec.), B.Sc.

Title of Paper : Foundations of Mathematics

Course Number : MAT231

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, A2, B2, ..., B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A (Answer ALL Questions in this Section)

Question A1 [20 Marks]

(a) Give clear definitions of the following terms:

(i) A *relation* $R : A \to B$. (2)

(ii) A *function* $f : A \to B$. (2)

(b) Let $X = \{1,2,3,4\}$. For each relation below, determine whether or not it is a function from X into X. Give reasons for your answers.

(i) $f = \{(2,3),(1,4),(2,1),(3,2),(4,4)\}$ (2)

(ii) $g = \{(3,1),(4,2),(1,1)\}$ (2)

(iii) $h = \{(2,1),(3,4),(1,4),(4,4)\}$ (2)

(c) (i) Define an *injection* $f : A \to B$. (2)

(ii) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x - 3$ is an injection. (2)

(d) Define the relation $\sim$ on $\mathbb{Z}$ as follows: For $m, n \in \mathbb{Z}$, $m \sim n$ if $m - n$ is divisible by 3. Prove that $\sim$ is an equivalence relation on $\mathbb{Z}$. (6)

Question A2 [20 Marks]

(a) Let p and q be propositions. Write down the valid arguments *modus ponens* and *modus tollens*. (4)

(b) Construct a truth table for the proposition $p \to (\neg p \vee q)$ (4)

(c) Use truth tables to prove the following: $\neg(p \wedge q) \equiv \neg p \vee \neg q$. (4)

(d) Let $D = \{1,2,3\}$ be the domain of discourse. Determine the truth values of the propositions

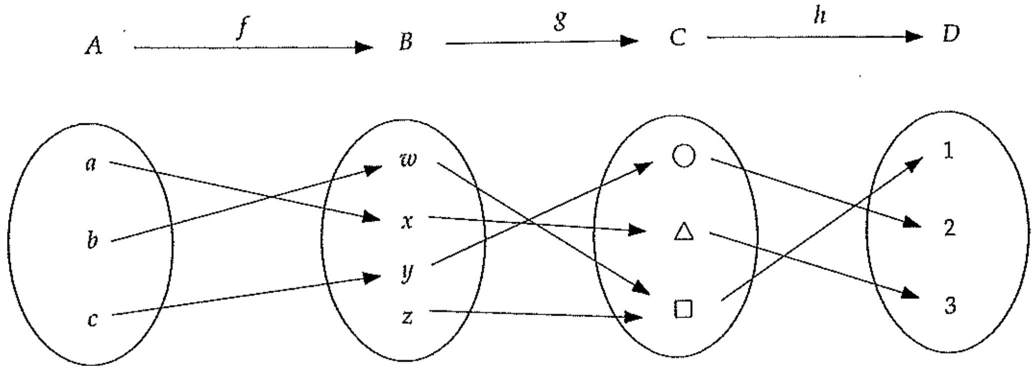
(i) $\forall x \forall y, x^2 + y^2 < 12$ (ii) $\exists x \forall y, x^2 < y + 1$ (iii) $\forall x \exists y, x^2 + y^2 < 12$ (2,3,3)

Question B5 [20 Marks]

(a) Find the domain of each function below.

(i) $f(x) = \dfrac{1}{x^2 - 4}$ (ii) $f(x) = \sqrt{x^2 - 4}$

(b) consider the functions $f, g,$ and h defined in the picture below.



Determine which functions are (i) injective, (ii) surjective, and (iii) invertible.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Fin
$(f \circ g)(x)$.

(d) Let $f : A \to B$ and $g : B \to C$ be surjective functions. Prove that $g \circ f : A \to C$
also a surjective function.

Question B6 [20 Marks]

(a) Prove: *For $a, b, c \in \mathbb{Z}$ with $b \neq 0$ and $c \neq 0$, if $b \mid a$ and $c \mid b$, then $c \mid a$.*

(b) Prove: *For an integer m, if m^2 is odd, then m is odd.*

(c) Prove: *For any integer n, $n^2 + n$ is even.*

(d) True or False? (If true, give a proof. If false, explain why.): *For all real numbers $x >$
$x > \dfrac{1}{x}$.*

Question B7 [20 Marks]

(a) Use mathematical induction to prove:

For all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

(b) Use strong induction to prove:

For all integers $n \geq 2$, either n is prime or n can be written as a product of prime numbers.

END OF EXAMINATION

Section B (Answer any three (3) Questions in this Section)

Question B3 [20 Marks]

(a) Let $U = \{1,2,3,\ldots,9\}$ and let $E = \{2,4,6,8\}$, $O = \{1,3,5,7,9\}$, and $P = \{2,3,5,7\}$. Find the following sets.

(i) $E \cap O$ (ii) $O \cap P$ (iii) $O \setminus P$ (iv) O^c

(b) Let $A = \{\triangle, \square, \bigcirc\}$. Write down $\mathscr{P}(A)$, the *power set* of the set A.

(c) Let A be any set and let $\emptyset$ be the emptyset. Show that $\emptyset \subseteq A$.

(d) Prove: *If* $A \subseteq B$, *then* $A \cap B = A$.

Question B4 [20 Marks]

(a) Use truth tables to show that the following argument is valid.

$$\begin{array}{rl} & p \to \neg q \\ & r \to q \\ & r \\ \hline \therefore & \neg p \end{array}$$

(b) Prove without truth tables: $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$.

(c) Negate each of the following statements.

(i) All UNESWA students live on campus.

(ii) The is a course such that for every UNESWA student, the student takes the course.

(iii) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - L| < \varepsilon$.

(iv) There is a real number x such that for all real numbers y, $x < y$.