UNIVERSITY OF ESWATINI

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RESIT EXAMINATION, 2019/2020

**BASS III, B.Ed (Sec.) III, B.Sc. III, B.Eng. III**

**Title of Paper** : Complex Analysis

**Course Number** : MAT313/M313

**Time Allowed** : Three (3) Hours

**Instructions**

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

**Special Requirements: NONE**

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

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## SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

**QUESTION A1 [*40 Marks*]**

a) Evaluate $\cos^{-1}(i)$ and leave your answer in the form $a+ib$. [5]

b) Find real constants $a$ and $b$ so that the function

$$f(z) = 3x - y + 5 + i(ax + by - 3)$$

is analytic. [5]

c) Evaluate $\int_C \frac{\sin^2(z) - 2z^3}{z^2 - 7z + 12}dz$ where $C$ is given by $|z| = 2$. [5]

d) Find the Maclaurin series of $\phi(z) = z\cos(z^2)$. [5]

e) Express $\int_0^{2\pi} \frac{5d\theta}{5 - 4\cos(\theta)}$ as a contour integral around the unit circle $|z| = 1$. [5]

f) Find the value of the residue at $z = 4$ for $f(z) = \frac{2z}{(z-4)(z-3)^2(z-1)}$. [5]

g) Express $z = \frac{1+3i}{-i+4}$ in the form $z = a + ib$. [5]

h) Use the precise definition of a limit to show that [5]

$$\lim_{z\to 2}(2iz - 2i) = 2i.$$

## SECTION B: ANSWER ANY *THREE* QUESTIONS

**QUESTION B2 [*20 Marks*]**

a) Consider the function $f(z) = \frac{1}{(z-1)(z-3)^2}$.

i) Locate and classify all singularities. [2]

ii) Find the values of the residues at all the singularities inside $|z| = 2$. [4]

iii) Hence evaluate $\int_C \frac{4}{(z-1)^3(z-3)^2}dz$, where $C$ is the circle defined by $|z| = 2$. [2]

b) Using Cauchy's Residue Theorem, evaluate [12]

$$\int_0^{2\pi} \frac{d\theta}{10 - 6\cos(\theta)}.$$

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**QUESTION B3** [*20 Marks*]

a) Let $f(z) = u(x,y) + iv(x,y)$, $\quad z_0 = x_0 + iy_0$, $\quad w_0 = u_0 + iv_0$. Prove that if [10]

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$$

and

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

then

$$\lim_{z\to z_0} f(z) = w_0$$

b) Show that $\alpha(x,y) = -e^{-x}\sin(y)$ is harmonic. Find the harmonic conjugate $\beta(x,y)$ and hence the analytic function $w(z) = \alpha(x,y) + i\beta(x,y)$. [10]

**QUESTION B4** [*20 Marks*]

a) Evaluate $\displaystyle\int_C |z|^2 dz$ where $C$ is parametrized by $x = t^2$, $y = \frac{1}{t}$ for $t \in [1,2]$. [8]

b) Evaluate $\displaystyle\int_C \frac{1}{z^3 + 2iz^2} dz$ where $C$ is $|z| = 1$. [4]

c) Evaluate $\displaystyle\int_C \frac{e^{z^2}}{(z-i)^3} dz$ where $C$ is parametrized by $|z - i| = 1$. [8]

**QUESTION B5** [*20 Marks*]

a) Determine if the sequence $\left\{\frac{(ni+2)^2}{n^2 i}\right\}$ for $n = 1, 2, \cdots$ converges or diverges. [4]

b) Determine whether the geometric series

$$\sum_{k=0}^{\infty} (1-i)^k$$

is convergent or divergent. [6]

c) Find the Laurent series that represents $f(z) = \dfrac{1}{z(z-1)}$ in the domain $|z| > 1$. [10]

**QUESTION B6** [*20 Marks*]

a) i) Show that $\cos^{-1}(z) = -i\ln\left(z + i\sqrt{1-z^2}\right)$ [10]

ii) Hence show that $\frac{d}{dz}\left(\cos^{-1}(z)\right) = \frac{-1}{\sqrt{1-z^2}}$ [4]

b) Find the principal value of $z = 2i^{-i}$ [6]

END OF EXAMINATION PAPER