University of Eswatini

Final Examination, December 2019

B.Sc III, B.A.S.S III, B.Ed III

Title of Paper : Real Analysis

Course Code : MAT331/M331

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

Question 1

(a) Define the following

(i) Let $S \subseteq \Re$ when do we say that S is bounded. [2]

(ii) Cauchy sequence. [2]

(iii) Uniformly continuous function. [2]

(b) (i) Prove that $|x| = max\{x, -x\}$. [3]

(ii) Evaluate $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1}$. [3]

(iii) The existence of the "inf" and the "sup" is an axiom of the real number system. It is not necessarily true that in any number system, a set that is bounded above has the least upper bound. Is this this statement true? Justify your answer. [4]

(iv) TRUE OR FALSE: The superimum (if it exists) of a set $S \subseteq \Re$, belong to S. Explain your answer. [4]

(v) Verify that the sequence $a_n = \frac{1}{n^2}, n \geq 1$ is monotone non-decreasing, monotone non-increasing or not monotone. [4]

(vi) Using the definition of limit (i.e., $\epsilon - \delta$), prove that
$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a.$$ [4]

(vii) Suppose $\sum a_n$ and $\sum b_n$ are positive term series with $a_n \leq b_n$ for all n. If $\sum b_n$ converges, show that $\sum a_n$ also converge. [4]

(viii) Test the series $\sum \frac{1}{2^n - n}$ for convergence. [4]

(ix) Does the $\lim_{x \to 1} \sin \frac{1}{x - 1}$ exist? If yes, find the limit. [4]

SECTION B: ANSWER ANY 3 QUESTIONS

Question 2

(a) For any two real numbers x, y show that $|x - y| \leq |x| + |y|$. [4]

(b) If A and B are bounded subsets of $\Re$, then prove that the set $A + B = \{x + y : x \in A \text{ and } y \in B\}$ is also bounded. [6]

(c) Find the *limsup* $a_n = (-1)^n + \frac{1}{n}$. [10]

Question 3

(a) By using $\epsilon - n$ definition prove that $\lim \frac{n}{n+1} = 1$. [10]

(b) Prove $\lim_{n \to \infty} \sqrt[n]{n} = 1$. [10]

Question 4

(a) $1 + a + \frac{a(a+1)}{1 \cdot 2} + \frac{a(a+1)(a+2)}{1 \cdot 2 \cdot 3} + \cdots$ $(a > 0)$ test for convergence using Gauss test. [10]

(b) Prove that a positive term series either converges or diverges to ∞. [10]

Question 5

(a) Show that $f(x) = \begin{cases} \frac{x^2-4}{x-2} & if\ x \neq 2 \\ 4 & if\ x = 2 \end{cases}$ is continuous at $x = 2$. [10]

(b) Prove that if a function f is uniformly continuous on an interval I, then it is continuous on I.

[10]

Question 6

(a) Prove that if a function is differentiable at a point then it is continuous at that point. [10]

(b) From the definition of the Riemann integral show that $\int_1^2 (2x+3) = 6$. [10]

End of Examination Paper