UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2019/2020

BASS IV, B.Ed (Sec.) IV, B.Sc. IV, B.Eng. III

Title of Paper : Partial Differential Equations

Course Number : MAT416/M415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

a) Consider the following equation with $-\infty < x < \infty$ and $t > 0$:

$$u_{tt} - 9u_{xx} = 0, \qquad u(x,0) = 3, \qquad u_t(x,0) = 12.$$

Determine $u(x,t)$. [7]

b) Express the partial differential equation

$$\begin{aligned} u_t &= u_{xx} \qquad 0 < x < 100, \qquad t > 0, \\ u(x,0) &= \sin(x), \qquad 0 \leq x \leq 100, \\ u(0,t) &= 100, \quad u(100,t) = 200, \end{aligned}$$

in the form such that the associated boundary conditions are homogeneous. [7]

c) Write down the ordinary boundary value problems for $X(x)$ and $T(t)$ that must be solved in order to obtain the solution of the wave equation

$$\begin{aligned} \phi_{tt} &= 9\phi_{xx}, \qquad 0 < x < \pi, \qquad t > 0, \\ \phi(x,0) &= 16\cos(x), \qquad 0 \leq x \leq \pi, \\ \phi_t(x,0) &= 0, \\ \phi(0,t) &= \phi(\pi,t) = 0. \end{aligned}$$

using the method of separation of variables. [7]

d) Show that [7]

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x,s) - u(x,0)$$

e) Determine the long term behaviour of the partial differential equation, [7]

$$u_t + u = 5, \qquad u(x,0) = 10$$

f) Write down the Laplacian in Cylindrical Polar Coordinates. [5]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

a) Consider the partial differential equation

$$u_t + u = e^t, \quad u(x,0) = e^{-x}$$

i) Determine $u(x,t)$ using direct substitution. [7]

ii) Determine the long term behaviour of the partial differential equation. [3]

b) Find the general solution of [10]

$$(y+u)u_x + yu_y = x - y,$$

using the method of characteristics.

QUESTION B3 [*20 Marks*]

a) Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$: [10]

$$u_{tt} - 9u_{xx} = 0, \quad u(x,0) - x = 0, \quad u_t(x,0) = e^{-x}.$$

Determine $u(x,t)$.

b) Consider the partial differential equation

$$u_{yy} + 5u_{xy} + 4u_{xx} + u_x + u_y = 0.$$

(i) Determine whether the given partial differential equation is hyperbolic, parabolic or elliptic. [2]

(ii) Express the given partial differential equation in canonical form. [8]

QUESTION B4 [*20 Marks*]

a) Show that the Laplacian of the function $u(x,y)$ in polar coordinates is given by [10]

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

b) Consider the Dirichlet problem of a sphere [10]

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) = 0,$$
$$u(1,\phi) = \phi, \quad 0 \le \phi \le \pi.$$

Solve the corresponding Euler-Cauchy equation obtained after separation of variables.

QUESTION B5 [*20 Marks*]

Consider the following equation [20]

$$\begin{aligned} u_t &= u_{xx} + 1, \quad 0 < x < \pi, \quad t > 0, \\ u(x,0) &= \sin(x), \quad 0 \le x \le \pi, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \end{aligned}$$

Determine the general solution of the equation using the method of separation of variables.

QUESTION B6 [*20 Marks*]

Consider the wave equation [20]

$$\begin{aligned} -16u_{xx} + u_{tt} &= e^{-2t}\cos(\pi x), \quad 0 \le x \le 1, \quad t \ge 0, \\ u(x,0) &= 0, \quad 0 \le x \le 1, \\ u_t(x,0) &= 0, \\ u(0,t) &= 0, \quad u(1,t) = 0. \end{aligned}$$

Using Laplace transform, show that the solution of the transformed equation is given by

$$U(x,s) = \frac{\cos(\pi x)}{(s+2)(s^2+16\pi^2)}$$

END OF EXAMINATION PAPER

$f(t)$	$\{f(t)\} = F(s)$
1	$\dfrac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$t^n \ (n = 0, 1, 2, \dots)$	$\dfrac{n!}{s^{n+1}}$
$t^x \ (x \geq -1 \in \mathbb{R})$	$\dfrac{\Gamma(x+1)}{s^{x+1}}$
$\sin kt$	$\dfrac{k}{s^2+k^2}$
$\cos kt$	$\dfrac{s}{s^2+k^2}$
e^{at}	$\dfrac{1}{s-a}$
$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$

$f(t)$	$f(t) = F(s)$
$\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
te^{at}	$\dfrac{1}{(s-a)^2}$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$
$e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$
$t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
$t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
$t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$
$t\cosh kt$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$
$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{s}$
$\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$