University of Eswatini

Resit/Supplementary Examination, 2019/2020

BASS IV, B.Ed (Sec.) IV, B.Sc. IV, B.Eng. III

Title of Paper : Partial Differential Equations

Course Number : MAT416

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

This examination paper should not be opened until permission has been given by the invigilator.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

(a) Consider the partial differential equation

$$u_{xx} - 2\sin(x)u_{xy} - \cos^2(x)u_{yy} - \cos(x)u_y + e^x = 0.$$

(i) Classify the partial differential equation by stating its order, linearity, homogeneity, and kind of coefficients.. [2]

(ii) Determine whether the given partial differential equation is hyperbolic, parabolic or elliptic. [2]

(iii) Determine the characteristic curves $\xi(x, y)$ and $\eta(x, y)$. [6]

(b) Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$:

$$p_{tt} - 4p_{xx} = 0, \quad 2p(x, 0) - \sin(x) = 0, \quad p_t(x, 0) = 4.$$

Determine $p(x, t)$. [7]

(c) Solve

$$xu_x + u_t = x, \quad u(x, 0) = u(0, t) = 0,$$

using the method Laplace transforms [7]

(d) Derive Parseval's identity theorem for the summability of the Fourier series coefficients of a function. [6]

(e) Consider the wave equation

$$\begin{aligned} \phi_{tt} &= c^2\phi_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ \phi(x, 0) &= 3\sin(x), \quad 0 \le x \le \pi, \\ \phi_t(x, 0) &= 0, \\ \phi(0, t) &= \phi(\pi, t) = 0. \end{aligned}$$

Write down the ordinary boundary value problems for $X(x)$ and $T(t)$ that must be solved in order to obtain the solution of the wave equation using the method of separation of variables. [5]

(f) Suppose that the temperature distribution in a rod of length π is given by $T(x, t)$. We assume that one end is kept at zero temperature and the other end ($x = \pi$) is insulated such that there is no heat flow. Write down a model that could be used to determine the temperature distribution $T(x, t)$, provided that the initial temperature distribution is given by $x^3 \sinh(\pi x)$. [5]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

a) Consider the partial differential equation

$$u_t + u = e^t, \qquad u(x,0) = e^{-x}$$

i) Determine $u(x,t)$ using direct substitution. [7]

ii) Determine the long term behaviour of the partial differential equation. [3]

b) Find the general solution of [10]

$$(y+u)u_x + yu_y = x - y,$$

using the method of characteristics.

QUESTION B3 [*20 Marks*]

Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$:

$$\rho_{tt} = v^2 \rho_{xx},$$
$$\rho(x,0) = \phi(x),$$
$$\rho_t(x,0) = \psi(x),$$

where v is a constant. Show that the solution of the wave equation is given by: [20]

$$\rho(x,t) = \frac{1}{2}\left(\phi(x+vt) + \phi(x-vt) + \frac{1}{v}\int_{x-vt}^{x+vt} \psi(\gamma)d\gamma\right)$$

QUESTION B4 [*20 Marks*]

(a) Use Laplace transforms to find a solution [12]

$$u_{xx} - u_t = \sin(\pi x), \qquad 0 \le x \le 3, \qquad t > 0,$$
$$u(x,0) = 0, \qquad 0 \le x \le 3,$$
$$u(0,t) = 0, \qquad u(3,t) = 0$$

(b) Using the fact that the Laplace transform of $u(x,t)$ with respect to the variable t is given by

$$\mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-st}u(x,t)dt \equiv U(x,s),$$

Show that $\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x,s) - u(x,0)$ [8]

QUESTION B5 [*20 Marks*]

Consider the radioactive decay problem given by

$$u_t = u_{xx} + 4e^{-x}, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = \sin(x), \qquad 0 \le x \le \pi,$$
$$u(0,t) = 0,$$
$$u(\pi,t) = 0,$$

Find $u(x,t)$ using the method of separation of variables. [20]

QUESTION B6 [*20 Marks*]

Consider the Dirichlet problem of a sphere of radius $r = a$.

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) = 0, \qquad 0 \leq r \leq a.$$

$$u(a,\phi) = f(\phi), \qquad 0 \leq \phi \leq \pi.$$

Find $u(r,\phi)$ using the method of separation of variables. [20]

END OF EXAMINATION PAPER

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$t^n \ (n = 0, 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$
$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
e^{at}	$\frac{1}{s-a}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$\frac{ae^{at}-be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$
$e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$
$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$
$t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$
$\frac{\sin at}{t}$	$\arctan \frac{a}{s}$
$\frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$