University of Eswatini

Main Examination, 2019/2020

B.Sc IV, BASS IV

Title of Paper : Abstract Algebra II

Course Number : M423/MAT423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

This examination paper should not be opened until permission has been given by the invigilator.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Let R be a ring. Define the following terms in R

i. a unity [2 marks]

ii. a zero divisor [2 marks]

iii. the characteristic [2 marks]

iv. Idempotent element of R [2 marks]

v. nilpotent element of R [2 marks]

(b) Consider $(R[x], +, \circ)$, where $\circ$ is a composition of polynomial.
By counterexample show that $(R[x], +, \circ)$ is not a ring.
(Hint: with $f(x) = x^2$, $g(x) = x$ and $h(x) = x$, check distributive law). [5 marks]

(c) Compute the evaluation homomorphism $\varphi_3[(x^4 + 2x)(x^3 - 3x^2 + 3)]$ in $\mathbb{Z}_6$. [5 marks]

(d) Let R be a ring. What is meant by

i. Subring of R [3 marks]

ii. an ideal of R [3 marks]

iii. a ring homomorphism $\beta : R \to R$ [3 marks]

(e) Show that $\mathbb{Z}_6$ is not an integral domain. [5 marks]

(f) If $1 - 2x,\ 1 + 2x^2 \in \mathbb{Z}_4[x]$. Evaluate $(1 - 2x)(1 + 2x^2)$ in $\mathbb{Z}_4[x]$. [6 marks]

SECTION B [*60 Marks*]: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Let S be the subset of all 2×2 real matrices $M_2(\mathbb{R})$ defined by
$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ a, b, c, d \in \mathbb{R}, \ a + c = b + d \right\}.$
Show that S is a subring of $M_2(\mathbb{R})$. [10 marks]

(b) Let R be an integral domain. If $f, g \in R[x]$ are both nonzero, then $fg \neq 0$.
Prove that $\deg(fg) = \deg(f) + \deg(g)$. [6 marks]

(c) Show that $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is an idempotent element in $M_2(\mathbb{R})$. [4 marks]

QUESTION B3 [*20 Marks*]

B3 (a) For any prime number p. Prove that $\mathbb{Z}_p$ is a field. [8 marks]

(b) Let $S = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : \ a, b \in \mathbb{Z} \right\}$ be a subset of $M_2(\mathbb{Z})$. Prove that S is an additive subgroup of $M_2(\mathbb{Z})$ and also S is a right-ideal but not a left ideal. [12 marks]

QUESTION B4 [*20 Marks*]

B4 (a) Prove that every field is an integral domain. [7 marks]

(b) Let R be a commutative ring with $\text{char}(R) = 2$. Define $\phi(x) = x^2$, for all $x \in \mathbb{R}$. Show that ϕ is a ring homomorphism. [6 marks]

(c) By Fermate's Little Theorem, evaluate

$$2^{20} + 3^{30} + 4^{40} + 5^{50} + 6^{60} \mod 7.$$

[7 marks]

QUESTION B5 [*20 Marks*]

B5 (a) Let I be an ideal of R and $a, b, c, d \in R$. If $a \equiv b(\mod I)$ and $c \equiv d(\mod I)$. Prove that $a + c \equiv b + d(\mod I)$. [4 marks]

(b) Find the quotient $q(x)$ and the remainder $r(x)$ when the polynomial $f(x) = x^3 + 2$ is divided by $2x + 2$ in $\mathbb{Z}_3[x]$. [7 marks]

(c) State Eisentein's criterion for irreducibility. [4 marks]

(d) Use Eisentein's criterion to show that $f(x) = 3x^4 - 10x^2 - 5x + 15$ is irreducible over $\mathbb{Q}$. [5 marks]

QUESTION B6 [*20 Marks*]

B6 (a) Prove that any prime element of an integral domain is irreducible. [6 marks]

(b) Define Unique factorisation domain (UFD). [5 marks]

(c) Find $d = \gcd(a, b)$ and x, y such that $d = ax + by$ if $a = 32 + 9i$ and $b = 4 + 11i$ in $\mathbb{Z}[i]$. [9 marks]

END OF EXAMINATION