University of Eswatini

Main Examination, 2019/2020

B.Sc IV and BASS IV

Title of Paper : Metric Space

Course Code : MAT434/M431

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Special requirements: None

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

A1. (a) Define a metric space. [4]

(b) Let $X = C[-1,1] := \{f : [-1,1] \to \mathbb{R} | \ f \text{ is continuous}\}$.
Define $\rho : X \times X \to \mathbb{R}^+ \ \forall \ f, g \in X$ by

$$\rho(f,g) = \int_{-1}^{1} |f(t) - g(t)| dt.$$

Compute

i. $f(t) = 5t$ and $g(t) = 0 \ \forall t \in [-1,1]$. [3]

ii. $f(t) = t$ and $g(t) = 2 - t \ \forall t \in [-1,1]$. [3]

(c) Consider $X = \mathbb{R}^2$. Define $d_1 : X \times X \to [0,\infty)$ by
$d_1(\bar{x},\bar{y}) = \sum_{i=1}^{2} |x_i - y_i|$ and$d_\infty(\bar{x},\bar{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.
If $\bar{x} = (-3,4)$ and $\bar{y} = (20,2)$. Find

i. $d_1(\bar{x},\bar{y})$ [3]

ii. $d_\infty(\bar{x},\bar{y})$. [3]

(d) Let (X,d) be a metric space and $A \subset X$.

i. Define the limit point of A in X. [3]

ii. Define the closure of A. [3]

(e) Consider the mapping $\rho : X \times X \to \mathbb{R}^+$ defined by $\rho(x,y) = |x - y|$.
Compute

i. $B_1(-1)$ [3]

ii. $\bar{B}_{\frac{3}{2}}(1)$ [3]

iii. $S_3(0)$. [3]

(f) Prove that $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (\frac{y}{2}, \frac{x}{2})$ is a contraction on $\mathbb{R}^2$ (with respect to the Euclidean metric). [5]

(g) Define a Homeomorphism. [4]

SECTION B: ANSWER ANY 3 QUESTIONS

B2. (a) Let $X = \mathbb{R}$(the reals) with metric ρ_0 defined by

$$\rho_0(x,y) = \begin{cases} 5, & x \neq y \\ 0, & x = y, \end{cases}$$

for arbitrary $x, y \in \mathbb{R}$. Describe the open balls:
(i) $B_6(-5)$ (ii) $B_{5.01}(-2)$ (iii) $B_2(3)$. [3,3,3]

(b) If X is any nonempty set, and $\rho : X \times X \to \mathbb{R}^+$ is a usual metric on X defined the function $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = \frac{\rho(x,y)}{1 + \rho(x,y)}$$

Prove that $d(x,y)$ is a metric on X. where $\rho(x,y) = |x - y|$. [11]

B3. (a) State the Banach contraction mapping principle. [6]

(b) Let $X = [4, \infty)$ with the usual metric for $\mathbb{R}$, and let $f : [4, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{2}\left(x + \frac{9}{x}\right) \quad \forall \ x \in [4, \infty).$$

Prove that

i. f is a contraction mapping on X. [8]

ii. What is the unique fixed point of f. [6]

B4. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 3x^2 + 7y^2 - 3x + 2y - 6$. prove that f is continuous at $(-1, 3)$. [10]

(b) Let $X = \mathbb{R}^2$ with the usual metric and $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^2$ is given by $x_n = \left(\frac{2n}{1+2n}, \frac{2}{1+n}\right)$ determine whether $\{x_n\}_{n=1}^{\infty}$ converges or not.
If it converges, find the limit and prove that the sequence indeed converges to the limit points. [10]

B5. (a) Let X be a nonempty set and suppose that (X, d) and (X, ρ) are metric spaces. When do we say that d and ρ are equivalent? [3]

(b) In $\mathbb{R}^n$ show that the metric $d_2(x,y) := \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$ and $d_\infty(x,y) : \max_{i \leq i \leq n} |x_i - y_i|$, are equivalent, where $x := (x_1, x_2, ..., x_n)$ and $y := (y_1, y_2, ..., y_n)$. [7]

(c) State the five consequences when a function $f : [a,b] \to \mathbb{R}$ is continuous. [10]

B6. (a) Prove that every subset of a metric space (X, ρ) is closed in X if and only if its complement is open in X. [7]

(b) Let (X, ρ_X) be a metric space and let (Y, ρ_Y) be a subspace of X. Let A be a subset of Y. Prove that A is closed in Y if and only if there exists a set F which is closed in X such that $A = Y \cap F$. [10]

(c) Let $X = \mathbb{R}$ (the reals) with the usual metric, and let $Y = [1, 2]$ be a subspace of X. Let $A = [1, \frac{3}{2})$. Show that $A = [1, \frac{3}{2})$ is not open in $X = \mathbb{R}$(with the usual metric). [3]

END OF EXAMINATION