University of Eswatini

Final Examination, December 2019

B.A.S.S. , B.Sc, B.Ed

Title of Paper : Dynamics II

Course Number : MAT455

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Non programmable calculators may be used (unless otherwise stated).
5. Special requirements: None.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: Answer All Questions

A1. (a) State whether true or false;

i. Under the Newtonian description, motion is described in special cartesian coordinates system.

ii. Lagrangian mechanics are independent of Newton's second law.

iii. The number of generalised coordinates required by a system is called the system's number of degrees of freedom.

iv. Constraints that can be expressed as a function in terms of the generalized coordinates and time are said to be non-holonomic.

v. Constraints involved in the motion of a particle on the surface of a sphere and the motion of gas molecules inside a container are examples of non-holonomic constraints.

[10]

(b) Prove the cancellation of dots property $\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}$. [5]

(c) The Lagrangian function of a system is given by

$$\frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}k(r-b)^2.$$

i. Determine the cyclic (ignorable) coordinates and find the generalized momenta conjugate to these coordinates. [4]

ii. Prove that the Hamiltonian of the system is given by

$$\frac{P_R^2}{2M} + \frac{P_r^2}{2\mu} + \frac{P_\theta^2}{2\mu r^2} + \frac{1}{2}k(r-b)^2.$$

[5]

iii. Determine Hamilton's equations of motion and prove that the equation of motion corresponding to r is

$$\mu(\ddot{r} - r\dot{\theta}^2) + k(r-b) = 0.$$

[5]

(d) Evaluate the poison bracket $[q^2p, qp]$. [6]

(e) Find the extremal of

$$I = \int_0^1 ((y'')^2 + y' + 3x^2)dx, \quad y(0) = 0, \quad y(1) = 1, \quad y'(0) = 1, \quad y'(1) = 1.$$

[6]

SECTION B: Answer Any THREE (3) Questions

B2. (a) Prove that if the transformation equations do not depend explicitly on time t and T is the kinetic energy, then

$$\sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T.$$

[10]

(b) The Lagrangian for a certain dynamical system is given by

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + t^2 - 2\dot{r}t\cos\theta + 2rt\dot{\theta}\sin\theta\right) - mg\left(\frac{1}{2}t^2 - r\cos\theta\right) - \frac{k}{2}(r-a)^2,$$

where r, θ are generalized coordinates, t is time and m, g and k are constants. Using the Lagrangian method, show that the equations of motion for the system are given by

$$\ddot{r} - r\dot{\theta}^2 = (g+1)\cos\theta - \frac{k}{m}(r-a),$$
$$\ddot{\theta} + 2\frac{\dot{r}\dot{\theta}}{r} + \frac{g+1}{r}\sin\theta = 0.$$

[10]

B3. Consider a particle moving on a real line. Let the dynamics of the particle be determined by the Hamiltonian,

$$H = \frac{q^4p^2}{2\mu} + \frac{\lambda}{q^2},$$

where μ and λ are real constants.

(a) Write down the Hamilton's equations of motion for the above system in their simplest form. [4]

(b) Find a Lagrangian for the system and write down the corresponding Lagrange's equation of motion. [16]

B4. (a) Prove that for a system whose dynamic behavior defined is by the Hamiltonian $H = H(q_j, p_j, t)$, the equation of motion for a dynamic variable $f(q_j, p_j, t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f,\ H].$$

[4]

(b) The Hamiltonian of a two-dimensional harmonic oscillator of unit mass is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2)$$

where ω is a constant. Given that $f = \omega q_1 \sin\omega t + p_1 \cos\omega t$, show that f is a constant of motion. [8]

(c) For what values of the constant parameters α and β is the transformation below canonical?

$$q = \beta P^{\alpha} \sin Q, \quad p = \beta P^{\alpha} \cos Q.$$

[8]

B5. (a) Find the curve $y(x)$ that minimizes the functional

$$\int_0^1 (y'^2 + y^2 + 2ye^{2x})dx, \quad y(0) = \frac{1}{3}, \quad y(1) = \frac{1}{3}e^2.$$

[10]

(b) Show that Euler-Lagrange equation for the functional

$$I = \int_{x=a}^{b} y\sqrt{1 + (y')^2}dx,$$

is given by

$$yy'' = 1 + (y')^2.$$

[10]

B6. (a) Show that if F has no explicit dependence on x (i.e $\partial F/\partial x = 0$) then $F = F(y, y')$ and the Euler-Lagrange's equation simplifies to the Beltrami-identity which is defined by the equation

$$F - y'\frac{\partial F}{\partial y'} = C,$$

where C is a constant. [10]

(b) Use the Beltrami identity to show that the extremum for the integral

$$I = \int_{x=0}^{a} \sqrt{\frac{1 + y'^2}{2y}}dx,$$

is given by

$$y' = \sqrt{\frac{c - y}{y}}.$$

[10]