UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2019/2020

M.Sc. in Mathematics

Title of Paper : Optimization

Course Number : MAT603

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions.
2. Answer ANY FIVE (5) questions.
3. Show all your working.
4. Start each new major question (Q1, Q2, ..., Q7) on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.
6. Some formulas are given on the last page.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1 [20 Marks]

(a) Give clear definitions of the following terms:

(i) A *convex set.* (2)

(ii) A *convex function.* (2)

(iii) A *concave function.* (2)

(b) Prove: *If f and g are both convex functions on a convex set S, then so is $f+g$.* (4)

(c) For each function below, determine whether it is concave, convex or neither on the given set.

(i) $f(x_1, x_2) = -x_1^2 - 3x_1x_2 - x_2^2$ on $\mathbb{R}^2$. (4)

(ii) $f(x_1, x_2) = x_1^2 + 2x_2^2$ on $\mathbb{R}^2$. (3)

(iii) $f(x_1, x_2) = -x_1^2 + x_1x_2 - 2x_2^2$ on $\mathbb{R}^2$. (3)

Question 2 [20 Marks]

(a) Consider the following unconstrained optimisation problem.

$$\text{maximise} \quad f(p_1, p_2) = p_1(60 - 3p_1 + p_2) + p_2(80 - 2p_2 + p_1) - 25p_1 - 72p_2.$$

Show that f is concave and find the values of p_1 and p_2 that maximise f. (6)

(b) Use the method of steepest descent to approximate the solution to the problem:

$$\text{minimise} \quad f(x_1, x_2, x_3) = (x_1 - 2)^2 - x_1 - x_2^2.$$

Begin at the point $\left(\frac{5}{2}, \frac{3}{2}\right)$. (7)

(c) Find all local maxima, local minima, and saddle points of

$$f(x_1, x_2) = x_1x_2 + x_2x_3 + x_1x_3.$$

(7)

Question 3 [20 Marks]

(a) Use Lagrange multipliers to solve the following problem.

$$\begin{aligned} \text{minimise} \quad & z = 2x^2 + y^2 - xy - 8x - 3y \\ \text{subject to:} \quad & 3x + y = 10. \end{aligned}$$

(10)

(b) Use K-T conditions to solve the following problem.

$$\begin{aligned} \text{maximise} \quad & z = x - y \\ \text{ssubject to:} \quad & x^2 + y^2 \leq 1. \end{aligned}$$

(10)

Question 4 [20 Marks]

(a) Use the simplex method to solve the following LP:

$$\begin{aligned} \text{maximise} \quad z = \; & 3x_1 + 2x_2 \\ \text{subject to:} \quad & 2x_1 + x_2 \leq 100, \\ & x_1 + x_2 \leq 80, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(10)

(b) Consider the following LP:

$$\begin{aligned} \text{maximise} \quad z = \; & 5x_1 - x_2 \\ \text{subject to:} \quad & 2x_1 + x_2 = 6, \\ & x_1 + x_2 \leq 4, \\ & x_1 + 2x_2 \leq 5, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Answer the following questions:

(i) Construct the initial Simplex tableau for the Big-M method. (4)

(ii) Determine the variable to enter the basis and the variable to leave the basis. (2)

(iii) What is the new basic feasible solution for the basis determined in (ii) above? Is this basic feasible solution optimal? (4)

Question 5 [20 Marks]

Consider the optimal control problem with state equation and cost function

$$\dot{x}_1 = -x_1 + u_1, \quad J = \int_0^{t_1} \left(k + \tfrac{1}{2}u_1^2\right) dt, \quad k > 0.$$

The initial and terminal states are $x_1(0) = X$ and $x_1(t_1) = 0$, respectively. The terminal time t_1 is free.

Use the PMP to find

(a) the optimal control,

(b) the optimal trajectory,

(c) the terminal time, and

(d) the optimal cost.

Question 6 [20 Marks]

Consider the controllable problem with state equations

$$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = 0, \quad |u_1| \leq 1.$$

(a) Find $C(t_1, 0)$, the set of all points controllable to the origin in time t_1. (6)

(b) Find $C(0)$, the set of all points controllable to the origin. (2)

(c) Find $\mathcal{R}(t_1, x^0)$, the set of reachable points from $x^0 = (p, q)^T$ in time t_1. (6)

(d) Write down the state equations for the time-reversed problem and verify that $\mathcal{R}(t_1, 0)$ for the time-reversed problem is the same as $C(t_1, 0)$ for the original problem. (6)

Question 7 [20 Marks]

(a) Consider the following LP:

$$\begin{array}{lrcrcl} \text{maximise} & z = -4x_1 & - & x_2 & & \\ \text{subject to:} & 4x_1 & + & 3x_2 & \geq & 6, \\ & x_1 & + & 2x_2 & \leq & 3, \\ & 3x_1 & + & x_2 & = & 3, \\ & & & x_1, x_2 & \geq & 0. \end{array}$$

After subtracting an excess variable e_1 from the first constraint, adding a slack variable s_2 to the second constraint and adding artificial variables a_1 and a_3 to the first and third constraints, it is found that in the optimal tableau, $x_{BV} = (x_2, x_1, e_1)^T$. Use the formulas to construct the optimal tableau of the LP. (8)

Hint: $$\begin{pmatrix} 3 & 4 & -1 \\ 2 & 1 & 0 \\ 1 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 3/5 & -1/5 \\ 0 & -1/5 & 2/5 \\ -1 & 1 & 1 \end{pmatrix}.$$

(b) Consider the following LP with its optimal tableau.

$$\begin{array}{lrcrcl} \text{maximise} & z = 3x_1 & + & 2x_2 & & \\ \text{subject to:} & x_1 & + & 2x_2 & \leq & 40, \\ & 2x_1 & + & x_2 & \leq & 50, \\ & & & x_1, x_2 & \geq & 0. \end{array}$$

z	x_1	x_2	s_1	s_2	rhs
1	0	0	1/3	4/3	80
0	1	0	−1/3	2/3	20
0	0	1	2/3	−1/3	10

Answer the following questions.

(i) Find the range of values of c_1 (the objective function coefficient of x_1) for which the current BV remains optimal. (4)

(ii) Find the range of values of b_1 (the right-hand side of the first constraint) for which the current BV remains optimal. (4)

(iii) A third activity x_3 is being considered. If $c_3 = 2$ and $a_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, determine if it is worth introducing this activity. (4)

SOME FORMULAS: $\bar{c}_j = c_{BV}B^{-1}a_j - c_j$, $\bar{b} = B^{-1}b$, $\bar{a}_j = B^{-1}a_j$, $\bar{z} = c_{BV}B^{-1}b$,

END OF EXAMINATION