University of Swaziland

Final Examination – November 2019

MSc in Mathematics

Title of Paper : Asymptotic Analysis
Course Number : MAT605
Time Allowed : Three (3) hours

Instructions:

1. This paper consists of 2 sections.
2. Answer ALL questions in Section A.
3. Answer ANY 3 (out of 5) questions in Section B.
4. Show all your working.
5. Begin each question on a new page.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A
Answer ALL Questions in this section

A.1 a. Consider the quadratic equation

$$x^2 + 3\varepsilon x - 4 = 0,$$

Find a 3-term perturbation solution of the form

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots,$$

valid when $\varepsilon << 1$. [15 marks]

b. Consider the initial value problem

$$\dot{y} + y = \varepsilon e^{-t}, \quad y(0) = 1.$$

i. Find the *exact* sloution of the problem. [5 marks]

ii. By letting

$$y(t;\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots, \quad \varepsilon << 1,$$

and substituting into the problem, obtain an expression for y_0, y_1 and $y_n, n \geqslant 2$. [10 marks]

c. Obtain a 2-term asymptotic approximation of the integral

$$\int_0^\infty e^{-\lambda t} t \sin\sqrt{t}\, dt$$

valid for large values of λ. [10 marks]

Section B

Answer ANY 3 Questions in this section

B.2 a. Consider the transcendental equation

$$x^2 + \varepsilon x = \cos \varepsilon x, \quad \varepsilon << 1.$$

Find a 3-term perturbation solution of the form

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots .$$ [10 marks]

b. Consider the nonlinear BVP

$$\ddot{y} - 2\dot{y} - \varepsilon y^2 = 0, \; y(0) = 1, \; \dot{y}(0) = 2.$$

Find a 2-term perturbation solution of the form

$$y(t;\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots, \quad \varepsilon << 1,$$

for the BVP. [10 marks]

B.3 Consider the quadratic equation

$$\varepsilon x^2 - 2x + 4 = 0.$$

Find a 3-term perturbation solution of the form

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots,$$

valid when $\varepsilon << 1$. [20 marks]

B.4 Consider the BVP

$$\varepsilon y'' + 2y' + y = 0, \;\; y(0) = 0, \; y(1) = 1,$$

where the parameter $\varepsilon << 1$. By assuming that a *boundary layer* exists at the $x = 0$ end, find

a. the leading order term of the *outer solution* [4 marks]

b. the *distinguished limit* and hence the rescaled inner variable [6 marks]

c. the leading order term of the *inner solution* [7 marks]

d. the leading order term of the *composite solution* [3 marks]

B.5 a. Prove that

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$ [7 marks]

b. Hence, prove that

$$\int_a^b f(t)e^{\lambda\varphi(t)}\mathrm{d}t \sim \left(\frac{2\pi}{-\lambda\varphi''(c)}\right)^{\frac{1}{2}} f(c)e^{\lambda\varphi(c)} \quad \text{as } \lambda \to \infty,$$

where c is a point in the interval $[a, b]$ where the value of φ is maximum. [13 marks]

B.6 a. Find a 2-term asymptotic approximation of of the integral

$$\int_{\lambda}^{\infty} \exp(-t^2)\mathrm{d}t$$

valid as $\lambda \to \infty$. [10 marks]

b. Derive *Sterling's Formula*

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n \to \infty.$$ [10 marks]

END OF EXAMINATION