UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2020

MSc

Title of Paper : POPULATION DYNAMICS & EPIDEMIOLOGY

Course Number : MAT606

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions.
2. Answer any FIVE (5) questions
3. Show all your working.
4. Start each new major question on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.
6. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1 [20 MARKS]

(a) State the Poincaré-Bendixson theorem. [2]

(b) Consider the dynamical system

$$\begin{aligned} \dot{x} &= y + ax(1 - b - x^2 - y^2), \\ \dot{y} &= -x + ay(1 - x^2 - y^2), \qquad 0 < a < 1, \quad 0 < b < 1. \end{aligned}$$

(i) Rewrite the system in polar coordinates by exploiting complex variables. [6]

(ii) Use the Poincaré-Bendixson theorem to prove the existence of a limit cycle for the flow. [4]

(iii) Show that for $b = 0$ there is only one limit cycle. [2]

(c) State and prove Bendixson's Negative criterion. [6]

QUESTION 2 [20 MARKS]

(a) (i) State the Dulac's criterion for the inexistence of closed orbits. [2]

(ii) By using a real valued function $g(x, y) = e^{ax+by}$ for some suitable constants a and b, show that the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - y + x^2 + y^2, \end{aligned}$$

has no limit cycles. [8]

(b) (i) By constructing a suitable Lyapunov function, determine the stability of the fixed points of

$$\begin{aligned} \dot{x} &= -x - 2y^2, \\ \dot{y} &= xy - y^3. \end{aligned}$$

[5]

(ii) Consider the population dynamical system below in which sheep $x(t)$ and goats $y(t)$ compete for resources in a closed ecosystem

$$\begin{aligned} \dot{x} &= x(3 - 2x - 2y), \qquad x \geq 0,\ y \geq 0 \\ \dot{y} &= y(2 - 2x - y). \end{aligned}$$

Sketch a phase portrait of the system. [5]

QUESTION 3 [20 MARKS]

A model for fishing in a lake, in nondimensional form, is geven by

$$\frac{df}{dt} = sf(1-f) - fn,$$
$$\frac{dn}{dt} = \alpha - n, \quad s > 0,\ \alpha > 0$$

where $f(t)$ and $n(t)$ respectively represent the number of fish and the number of fishermen.

(a) Suggest a plausible interpretation for each of the terms in the model. [4]

(b) Determine the equilibrium solutions of the model and determine their stability. Interprete your results in terms of the long term dynamics of fishing in the lake. [10]

(c) Sketch a phase portrait of the model. Explain how the qualitative nature of the phase portrait depends on the model parameters. [6]

QUESTION 4 [20 MARKS]

A model for the interaction of a predator v and prey u is described by

$$\frac{du}{dt} = Ru\left(1 - \frac{u}{k}\right) - A_1\frac{u}{A_2 + u}v,$$
$$\frac{dv}{dt} = -B_1 v + B_2\frac{u}{A_2 + u}v,$$

where R, k, A_1, A_2, B_1, and B_2 are strictly positive constants.

(a) Explain the biological meaning of each of the four terms in the model. [4]

(b) Use the variable transformations

$$x = \frac{u}{k}, \quad y = \frac{A_1}{kR}v, \quad T = Rt,$$

to show that the system can be written in dimensionless form

$$\frac{dx}{dT} = x(1-x) - \frac{x}{\theta + x}y,$$
$$\frac{dy}{dT} = -ay + c\frac{x}{\theta + y}y.$$

[6]

(c) (i) Show that the model always has a critical point at $(0,0)$ and $(1,0)$ and determine the nature of these critical points. [8]

(ii) Show that the model has a third critical point $(\bar{x}, \bar{y})$ if and only if $\theta < \frac{c-a}{a}$. [2]

QUESTION 5 [20 MARKS]

The following diagram represents the dynmics of SIR epidemic model without births or deaths.

$$S \xrightarrow{\beta} I \xrightarrow{\gamma} R$$

where β and γ are positive constants.

(a) Write down the set of equations to describe these dynamics and explain why the system can be fully described using two of the equations. [6]

(b) (i) Define basic reproductive ratio, R_0, in words. [2]

(ii) Derive the expression for R_0 in terms of your model parameters. [4]

(c) Deermine the maximum number of infectives ($I_{\max}$) [8]

QUESTION 6 [20 MARKS]

Suppose a population structure according to a disease is described with the following equations

$$\frac{dS}{dt} = -\beta IS + \gamma R,$$
$$\frac{dI}{dt} = \beta SI - \nu I,$$
$$\frac{dR}{dt} = \nu I - \gamma R.$$

(a) Describe all the terms in the model. [6]

(b) What is R_0 in this situation? [2]

(c) By letting $N = S + I + R$, reduce the system to three equations to a two equations model, determine all the equilibria and investigate the stability of the disease free equilibrium point. [10]

(d) Give an example of a disease which can have the same dynamics. [2]

QUESTION 7 [20 MARKS]

Given the Fisher equation

$$\frac{\partial u}{\partial t} = Au\left(1 - \frac{u}{k}\right) + D\frac{\partial^2 u}{\partial x^2}$$

where u represents a population density, A and D are positive constants and k is the populatiion carrying capacity.

(a) By re-scaling the Fisher equation using

$$U = \frac{u}{k}, \quad T = At, \quad X = x\left(\frac{A}{D}\right)^{\frac{1}{2}},$$

show that

$$\frac{\partial U}{\partial T} = U(1-U) + \frac{\partial^2 U}{\partial t^2}. \qquad (\Psi)$$

[6]

(b) By assuming a wave solution of the form $U(X - cT)$, convert (Ψ) into a second order ordinary differential equation with an independednt variable $z = X - cT$. [4]

(c) Use the substitution $V = U'$ to convert the second order ordinary differential equation into a pair of first order ordinary differential equations. Hence find the co-ordinates of the fixed points of the system and investigate their stability. [10]

——— The end ———