UNIVERSITY OF ESWATINI

DECEMBER 2019 MAIN EXAMINATION

MSc in Mathematics

Title of Paper : Spectral Methods for Differential Equations

Course Number : MAT607

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions.
2. Answer ANY FOUR (4) questions.
3. Show all your working.
4. Start each new major question on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1 [*25 Marks*]

The solution of the differential equation

$$y''(x) - 3y'(x) + 2y(x) - x = 0$$

with boundary conditions

$$y(0) = 1, \text{ and } y(1) = 0$$

can be approximated by the polynomial

$$Y(x) = c_0 + c_0 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

Use spectral collocation points with equally spaced collocation points to show that matrix equation that results from the collocation process is [25 Marks]

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & -\frac{5}{2} & \frac{5}{8} & \frac{31}{32} & \frac{73}{128} \\ 2 & -2 & -\frac{1}{2} & 1 & \frac{13}{8} \\ 2 & -\frac{3}{2} & -\frac{11}{8} & \frac{9}{32} & \frac{297}{128} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \\ 0 \end{bmatrix}$$

QUESTION 2 [*25 Marks*]

Consider the Blassius boundary layer flow equation

$$f'''(x) + f(x)f''(x) = 0$$

whose boundary conditions are

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1$$

(a) Derive the quasi-linearisation method scheme that can be used to iteratively solve the Blassius equation [10 Marks]

(b) Illustrate how the matrix approach of the spectral collocation can be applied on the quasi-linearisation method and boundary conditions with the transformations

$$f^{(n)}(x_i) = \sum_{k=0}^{N} \mathbf{D}_{i,k}^{(n)} f(z_k) = \mathbf{DF}, \quad i = 0, 1, 2 \ldots, N$$

where $\mathbf{D}$ is the differentiation matrix and $\mathbf{F}$ is the vector of unknowns at the so-called collocation points $z_i = \cos\left(\frac{\pi i}{N}\right)$. [15 Marks]

QUESTION 3 [*25 Marks*]

Consider the following linear partial differential equation with boundary conditions and initial conditions

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} - 2x\frac{\partial y}{\partial x} - y$$

$$y(-1,t) = e^t, \quad y(1,t) = e^t, \quad \text{and} \quad y(x,0) = e^{x^2-1}$$

with exact solution $y(x,t) = e^{t+x^2-1}$.

Is approximating the solution of the differential equation, consider three equally spaced nodes x_0, x_1, x_2 in the space variable x and two nodes t_0, t_1 in the time variable t and the approximating function.

$$Y(x,t) = c_{0,0} + c_{1,0}x + c_{2,0}x^2 + c_{0,1}t + c_{1,1}tx + c_{2,1}tx^2.$$

Show that the collocation process leads to the matrix equation

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & \frac{1}{10} & -\frac{1}{10} & \frac{1}{10} \\ 1 & 0 & -2 & \frac{1}{10} & 0 & -\frac{1}{5} \\ 1 & 1 & 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} c_{0,0} \\ c_{1,0} \\ c_{2,0} \\ c_{0,1} \\ c_{1,1} \\ c_{2,1} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-1} \\ 1 \\ e^{1/10} \\ 0 \\ e^{1/10} \end{bmatrix}$$

QUESTION 4 [*25 Marks*]

Consider the linear system of ordinary differential equations

$$\frac{d^2u}{dx^2} - u - \frac{dw}{dx} = 0$$
$$\frac{d^2w}{dx^2} + \frac{dw}{dx} + \frac{du}{dx} + w = 0$$

subject to the following boundary conditions

$$u(a) = 0, \quad w'(a) = 1$$
$$u'(b) = -1 \quad w(b) = -1$$

Use the matrix based spectral collocation method with to illustrate how the linear system can be reduced to a matrix system of the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

In your illustration, give definitions of the matrices and vectors and demonstrate how the boundary conditions can be imposed on the matrices and vectors. [25 Marks]

QUESTION 5 [*25 Marks*]

Consider the linear ordinary differential equation

$$(1+x^2)\frac{d^2u}{dx^2} + 4x\frac{du}{dx} + 2u = 0$$

with boundary conditions

$$u(-1) = \frac{1}{2}, \quad u(1) = \frac{1}{2}.$$

and exact solution $u(x) = \frac{1}{1+x^2}$

Give a sketch of the Matlab code that can be used to solve the differential equation using a function, say *cheb.m* for invoking the collocation points x and differentiation matrix D. Your code sketch must include a line for plot the exact vs approximate solution and residual error profile. [25 Marks]

QUESTION 6 [*25 Marks*]

Consider the regular eigenvalue problem

$$y''(x) + \lambda y(x) = 0, \qquad x \in (0, 1)$$

subject to the boundary conditions

$$y'(0) = 0, \qquad y(1) = 0$$

Describe how the spectral quasi-linearisation method can be used to solve the eigenvalue problem [25 Marks]

END OF EXAMINATION PAPER