UNIVERSITY OF ESWATINI

MAIN EXAMINATION 1, 2019/2020

M.Sc. Mathematics 1

Title of Paper : STOCHASTIC DIFFERENTIAL EQUATIONS

Course Number : MAT 632

Time Allowed : Three (3) Hours

Instructions:

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Start each new major question (A1-A5, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.
6. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1.

(a.) Define a stochastic process. [2 marks]

(b.) Evaluate the stochastic integral $I = \int_0^t s^2 B^3 dB(s)$. [6 marks]

QUESTION A2.

(a.) Define a σ-algebra over a non empty set Ω. [3 marks]

(b.) Suppose $G_1, G_2, G_3, ..., G_n$ are disjoint subsets of Ω such that $\Omega = \bigcup_{i=1}^n G_i$. Prove that a family G consisting of $\emptyset$ and all unions of $G_1, G_2, G_3, ..., G_n$ constitute a $\sigma-$ algebra on Ω. [5 marks]

QUESTION A3.

(a.) Define a Brownian Motion. [2 marks]

(b.) Evaluate:

i. $E[B^4]$ [3 marks]

ii. $E[B^{24}]$ [3 marks].

QUESTION A4.

Use Ito's formula to write $X(t)$ in the form

$$dX(t) = u(t, \omega)dt + v(t, \omega)dB(t)$$

(i.) $X(t) = t^7 B^2(t)$. [4 marks]

(ii.) $X(t) = t^3 + e^{B(t)}$. [4 marks]

QUESTION A5.

Given a Brownian motion $\{B_t\}_{t \geq 0}$. Show that the increments of $\{B_t\}_{t \geq 0}$ are independent. [8 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2

a.) State the Martingale representation theorem. [4 marks]

(b.) Prove that if

$$dZ(t) = Z(t)\theta(t,\omega)dB(t)$$

then $Z(t)$ is a martingale for all $t \leq T$ provided that
$Z(t)\theta_k(t,\omega) \in \nu(0,T) \quad 1 \leq k \leq n$. [16 marks]

QUESTION B3

a.) Define an elementary function. [2 marks]

b.) List three (3) real valued functions that are elementary functions. [5 marks]

c.) State and prove the Ito isometry for elementary and bounded function $\phi(t,\omega)$. [13 marks]

QUESTION B4.

(a.) Define a Wiener process. [4 marks]

(b.) Given a stochastic process $\{X(t)\}_{t\geq 0}$ whose changes is described by

$$\frac{dX(t)}{dt} = a(t,X(t)) + b(t,X(t)).''noise.''$$

where $a(.)$ and $b(.)$ are functions. Given that $W(t) \sim B(t)$, Construct a discrete time solution for $X(t)$ in the interval $[0,t]$. [16 marks]

QUESTION B5.

(a.) Solve the stochastic differential equation for $\alpha \in \Re$

$$dX(t) = \alpha X(t)dt + \rho X(t)dB(t); \ \rho, \alpha \in \Re. \quad [10 marks]$$

(b.) Evaluate:

(i.) $E[X]$. [5 marks]

(ii.) $\sigma^2(X)$. [5 marks]

QUESTION B6.

(a.) Solve the Ornstein-Uhlenbeck equation

$$dX(t) = \alpha X(t)dt + \rho dB(t); B(0) = 0, \rho \in (0,1), \alpha \in \Re. \qquad [10 marks]$$

(b.) Evaluate:

(i.) $E[X]$. [5 marks]

(ii.) $\sigma^2(X)$. [5 marks]

END OF EXAMINATION