UNIVERSITY OF ESWATINI

FINAL SEMESTER I EXAMINATION, 2019/2020

M.Sc. Mathematics

Title of Paper : Advanced Applied Analysis

Course Number : MAT633

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions. Answer ANY FIVE (5) questions.
2. You can answer questions in any order.
3. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1 [*20 Marks*]

1 (a) Define what is meant by saying that the functions f_n *converge uniformly and pointwisely* to f on the interval $[a, b]$ as $n \to +\infty$. [5 marks]

(b) State Uniformly Cauchy Criterion Theorem. [5 marks]

(c) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on a set D, if $f_n \to f$ uniformly on D. Prove that f is continuous on D. [5 marks]

(d) Show that $f_n(x) = e^{x\frac{1+n}{n}}$ converge on $\mathbb{R}$ uniformly. [5 marks]

QUESTION 2 [*20 Marks*]

2 (a) Suppose X is a vector space over the field $\mathbb{F}$. Write down the conditions for X to be a normed vector space together with the real valued function $||.|| : X \to [0, \infty)$. [5 marks]

(b) Prove that the classical space $\ell_p(1 \leq p < \infty)$ is complete. [12 marks]

(c) A measure space (X, Σ, μ) is complete if? [3 marks]

QUESTION 3 [*20 Marks*]

3 (a) Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous function. Suppose that $\{f_n\}$ converges uniformly to some $f : [a, b] \to \mathbb{R}$ on $[a, b]$. Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

[8 marks]

(b) Let $f_n : [0, 1] \to \mathbb{R}$, where $n \in \mathbb{R}$, be defined by

$$f_n(x) = \frac{n + (\sin(e^x))^n}{2n + x^3}, \quad x \in [0, 1].$$

Find $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx$. [6 marks]

(c) By Weierstrass M-Test, prove that the series

$$\sum_{n=1}^{\infty} \frac{1 + n\sin(nx)}{n^{4-\cos(nx)}}$$

is uniformly convergent on $[0, 2\pi]$. [6 marks]

QUESTION 4 [*20 Marks*]

4 (a) Let $X = C[0, 1]$. Define $\langle .,. \rangle : X \times X \to \mathbb{C}$ for each $f, g \in X$ the inner product on X by $\langle f, g \rangle = \int_0^1 \overline{f(t)}g(t)dt$, where $\overline{f(t)}$ is the conjugate of $f(t)$. Compute $\langle .,. \rangle$, when $f(t) = g(t) = 1 + it$. [5 marks]

(b) Let X be inner product space. Suppose $x, y \in X$ such that

$$||x|| = \sqrt{17}, \ ||x + y|| = 4 \ \text{ and } \ ||x - y|| = 6.$$

Find $||y||$. [5 marks]

(c) Let $X = \mathbb{R}^3$ for any $x, y \in \mathbb{R}^3$ define the inner product and norm on $\mathbb{R}^3$ by

$$\langle x, y\rangle = x^T y \quad \text{and} \quad ||x||_2 = \sqrt{\sum_{i=1}^{3} x_i^2}$$

respectively. Given a set of three linearly independent vectors in $\mathbb{R}^3$

$$x^{(1)} = \begin{pmatrix}1\\0\\1\end{pmatrix}; \quad x^{(2)} = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad x^{(3)} = \begin{pmatrix}2\\1\\0\end{pmatrix}.$$

Using Gram Schmidt procedure, generate an orthonormal set. [10 marks]

QUESTION 5 [*20 Marks*]

5 (a) Let $\mathcal{F}$ be a family of functions from a metric space (X, d) to a metric space (Y, d). Define what is meant by saying that the functions $\mathcal{F}$ is *equicontinuous and uniformly equicontinuous* on X. [4 marks]

(b) Prove that every equicontinuous family of functions from a compact metric space to a metric space is uniformly equicontinuous. [8 marks]

(c) Let $\mathcal{F}$ be the subset of $C[0,1]$ that consists of functions f of the form

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad \text{with} \quad \sum_{n=1}^{\infty} n|a_n| \leq 1.$$

The series defining f converges uniformly. By Arzelà-Ascoli, prove that $\mathcal{F}$ is a compact subset of $C[0,1]$. [8 marks]

QUESTION 6 [*20 Marks*]

6 (a) Let (X, d) be a metric space and $f : X \to X$ a mapping on X.

i. Define what it means for f to be a contraction. [4 marks]

ii. Every continuous map is a contraction map. Yes or No? [2 marks]

iii. State the Contraction Mapping Principle. [6 marks]

(b) Consider the nonlinear, scalar ODE given by

$$\dot{u}(t) = \sqrt{a(t)^2 + u(t)^2},$$
$$u(0) = u_0,$$

where $a : \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove that ODE have a solution. [8 marks]

QUESTION 7 [*20 Marks*]

7 (a) Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal sequence in an infinite dimensional Hilbert space.

i. State Bessel's inequality. [4 marks]

ii. What happen to Bessel's inequality when $\{\phi_n\}_{n=1}^{\infty}$ is a complete orthonormal sequence? [3 marks]

(b) Let(X, Σ, μ) be a measure space and f a nonnegative measurable function. Define a Lebesgue integral of f with respect to μ denoted by $\int_X f d\mu$. [4 marks]

(c) Let $1 < p, q < \infty$ be conjugate exponents, (X, Σ, μ) a measure, if $f \in L^p(X, \mu)$, $g \in L^q(X, \mu)$ with $p \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that

$$fg \in L^1(X, \mu) \quad \text{and} \quad \int_X |fg| d\mu \leq ||f||_p ||g||_q.$$

[9 marks]

END OF EXAMINATION